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NOTES

Class:- B.sc 3rd sem (NEP)

Sub:- OPERATION RESEARCH

Type your text



What is Operations Research?

1. OR is an art of winning the war without actually fighting it.
2. OR is the art of finding best answers where worst exist. - T.L. Saaty (1958)
3. OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. - Morse and Kimball (1946)
4. OR is a scientific method of providing executive with an analytical and objective basis for decisions.
5. OR is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problem. - Churchman, Acoff, Arnoff (1957)

Applications of OR

1. Finance, Budgeting and Investments:
 - (a) Claims and complaint procedures.
 - (b) Credit policies, credit risks and delinquent

account procedures.
(c) Cash-flow analysis, long range capital requirements, dividend policies, investment portfolios.

2. Purchasing, Procurement and Exploration:

- (a) Rules for buying, supplies and ~~states~~ state on varying prices.
- (b) Determination of quantities and timing of purchases.
- (c) Bidding policies.
- (d) Strategies for exploration and exploitation of raw material sources.
- (e) Replacement policies.

3. Production Management:

- (i) Physical: (a) location and size of warehouse, distribution centres and retail outlets.
(b) Distribution policy.
- (ii) Facilities: (a) Numbers and location of factories, warehouses, hospitals etc.
(b) Loading and unloading facilities for railroads and trucks determining the transport schedule.

- (iii) Manufacturing: (a) Production scheduling and sequencing.
(b) Stabilization of production and employment training, layoffs and optimum product mix.

- (iv) Maintenance and Project Scheduling: (a) Maintenance policies and preventive maintenance.
(b) Maintenance crew size.
(c) Project scheduling and allocation of resources.

4. Marketing:

- (a) Product selection, timing, competitive actions.
- (b) Number of salesman, frequency of calling on accounts per cent of time spent on prospects.
- (c) Advertising media with respect to cost and time.

5. Personnel Management:

- (a) Selection of suitable personnel on minimum salary.
- (b) Mixes of age and skills.
- (c) Recruitment policies and assignment of jobs.

6. Research and Development :

- Determination of the areas of concentration of research and development.
- Project selection.
- Determination of time cost trade-off and control of development projects.
- Reliability and alternative design.

We can conclude that OR can be widely used in taking timely management decisions and also used as a corrective measure.

Main Characteristics of OR

- Inter-disciplinary team approach : In OR, the optimum solution is found by a team of scientists selected from various disciplines such as mathematics, statistics, economics, engineering, physics etc.

Ex : An OR team required for a big organization may include a statistician, an economist, a mathematician, one or more engineers, a psychologist, and some supporting staff like computer programmers, etc.

- Wholistic approach to the system : The most of the problems tackled by OR have the characteristic that OR tries to find the best (optimum) decisions relative to largest possible portion of the total organization. The nature of organization is essentially immaterial.

- Imperfectness of solutions : By OR techniques, we cannot obtain perfect answers to our problems but, only the quality of the solution is improved from worse to 'bad' answers.

- Use of scientific research : OR uses techniques of scientific research to reach the optimum solution.

- To optimize the total output : OR tries to optimize total return by maximizing the profit and minimizing the cost or loss.

MAIN PHASES of OR Study

Phase I : Formulating the Problem :- Before proceeding to find the solution of a problem, first of all one must be able to formulate the problem in the form of an appropriate model.

- To do so, following information is required :
- Who has to take the decision?
 - What are the objectives?

- (iii) What are the ranges of controlled variables?
- (iv) What are the uncontrolled variables that may affect the possible solutions?
- (v) What are the restrictions or constraints on the variables?

Phase II: Constructing a mathematical model:
The second phase of the investigation is concerned with the reformation of the problem in an appropriate form which is convenient for analysis. The most suitable form for this purpose is to construct a mathematical model representing the system under study. It requires the identification of both static and dynamic structural elements.

A mathematical model should include the following three important basic factors:

- (i) Decision variables and parameters.
- (ii) Constraints or Restrictions
- (iii) Objective function.

Phase III: Deriving the solutions from the model
This phase is devoted to the computation of those values of decision variables that maximize (or minimize) the objective function. Such solution is called an optimal solution which is always in the best interest of the problem under consideration.

Phase IV: Testing the model and its solution (updating model)

After completing the model, it is once again tested as a whole for the errors if any. A model may be said to be valid if it can provide a reliable prediction of the system's performance.

Phase V: Controlling the solution

This phase establishes controls over the solution with any degree of satisfaction. The model requires immediate modification as soon as the controlled variables change significantly, otherwise the model goes out of control. As the conditions are constantly changing in the world, the model and the solution may not remain valid for a long time.

Phase VI: Implementing the solution

Finally, the test results of the model are implemented to work. This phase is primarily executed with the cooperation of Operations Research experts and those who are responsible for managing and operating the systems.

Scope of Operations Research

1. In Agriculture : With the explosion of population and consequent shortage of food, every country is facing the problems of -
 - (i) optimum allocation of land to various crops in accordance with the climatic conditions.
 - (ii) optimum distribution of water from various resources like canal for irrigation purposes.

Thus there is a need of determining best policies under the prescribed restrictions. Hence a good amount of work can be done in this direction.

2. In Finance : In these modern times of economic crisis, it has become very necessary for every government to have a careful planning for the economic development of her country.

OR-techniques can be fruitfully applied:

- (i) to maximize the per capita income with minimum resources.
- (ii) to find out the profit plan for the company.
- (iii) to determine the best replacement policies.

3. In Industry : OR is useful to the industry ~~Distri~~ Director in deciding optimum allocation of various limited resources such as men, machines, materials, money, time etc to arrive at the optimum decision.

4. In Marketing : With the help of OR techniques a Marketing Administrator can decide -

- (i) when to distribute the products for sale so that the total cost of transportation etc is minimum.
- (ii) the minimum per unit sale price.
- (iii) the size of the stock to meet the future demand.
- (iv) how to select the best advertising media with respect to time, cost, etc.
- (v) how, when, and what to purchase at the minimum possible cost?

5. In Personnel Management : A personnel manager can use OR techniques :

- (i) to appoint the most suitable persons on minimum salary.
- (ii) to determine the best age of retirement for the employees.
- (iii) to find out the number of persons to be appointed on full time basis when the workload is seasonal (not continuous).

6. In Production Management : A production Manager can use OR techniques :

- (i) to find out the number and size of the items to be produced.
- (ii) In scheduling and sequencing the production run by proper allocation of machines.
- (iii) In calculating the optimum product mix.
- (iv) To select, locate and design the sites for the production plants.

7. In L.I.C. :- OR approach is also applicable to enable the L.I.C. officers to decide:
- (i) what should be the premium rates for various modes of policies.
 - (ii) how best the profits could be distributed in the case of with profit policies?

In Conclusion, we can say, whenever there is a problem, there is OR.

Linear Programming Problems [LPP]

The general LPP is for optimizing (maximizing / minimizing) a linear function of variables called the objective function subject to a set of linear equations and/or inequalities called the constraints or restrictions.

Eg: $\text{Max } Z = 2x_1 + 3x_2$ } objective functions
s.t

$x_1 + x_2 \leq 3$ } set of constraints
 $2x_1 + 5x_2 \geq 7$

$x_1, x_2 \geq 0$ } non-negative restriction

Formulations of LPP problems

Ques 1

A firm manufactures two types of products A and B and sell them at a Profit of ₹ 2 on type A and ₹ 3 on type B. Each product is processed from two machines G & H. Type A requires 1 min of processing time on G and 2 min on H. Type B requires 1 min on G and 1 min on H. The machine G is available for not more than 6 hr 40 min, while machine H is available for 10 hr during any working day. Formulate the problems as a LPP.

Solⁿ:

Let x_1 be the no. of products of Type A
& x_2 be the no. of products of Type B

(Total Profit) $\text{Max } Z = 2x_1 + 3x_2$ } objective function

$G = 400 \text{ min}$

$H = 600 \text{ min}$

s.t, $x_1 + x_2 \leq 400$ } set of constraints
 $2x_1 + x_2 \leq 600$

$x_1, x_2 \geq 0$ } non-negative restrictions.

Ques 2

A company produces two types of Hats. Each hat of the first type requires twice as much labour time as the second type. If all hats are of the second type only, the company can produce a total of 500 hats.

A day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profits per hat are Rs. 8 for type A and Rs. 5 for type B, formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit.

Solⁿ: Let x_1 be the hats of Type A.
& x_2 be the hats of Type B.

$$\text{Max } z = 8x_1 + 5x_2 \quad \left. \begin{array}{l} \text{S.t} \\ 2x_1 + x_2 \leq 500 \\ x_1 \leq 150 \\ x_2 \leq 250 \\ x_1, x_2 \geq 0 \end{array} \right\} \text{Objective function}$$

$$\left. \begin{array}{l} 2x_1 + x_2 \leq 500 \\ x_1 \leq 150 \\ x_2 \leq 250 \end{array} \right\} \text{Set of constraints}$$

$$x_1, x_2 \geq 0 \quad \left. \begin{array}{l} \text{non-neg. restrictions.} \end{array} \right\}$$

Ques 3 The manufacturer of patent medicines is proposed to prepare a production plan for medicines A and B. There are sufficient ingredients available to make 20,000 bottles of medicine A and 40,000 bottles of medicine B, but there are only 45,000 bottles into which either of the medicines can be filled. Further it takes three hours to prepare enough material to fill 1000 bottles of medicine A and one hour to prepare enough material to fill 1000 bottles of medicine B, and there are 66 hours

available for this operation. The profit is Rs 8 per bottle for medicine A and Rs 7 per bottle for medicine B.

- (i) Formulate this problem as an L.P.P.
(ii) How the manufacturer schedule his production in order to maximize profits.

Solⁿ: Let there be x_1 thousand bottles of medicine A and x_2 thousand bottles of medicine B

Profit on thousands bottle of x_1 is at the rate of Rs 8 and on thousands bottle of x_2 is at the rate of Rs 7.

$$\Rightarrow \text{Max } z = 8 \times 1000 x_1 + 7 \times 1000 x_2$$

$$\therefore \text{Max } P = 8000 x_1 + 7000 x_2 \quad \left. \begin{array}{l} \text{Objective fn} \end{array} \right\}$$

Time required to fill x_1 thousand bottles of medicine A will be $3x_1$ and x_2 thousand bottles of medicine B will be x_2 .
But, total $\Rightarrow 3x_1 + x_2$

Now, total time available for this operation is 66 hrs.

$$\left. \begin{array}{l} 3x_1 + x_2 \leq 66 \\ x_1 + x_2 \leq 45 \\ x_1 \leq 20 \\ x_2 \leq 40 \end{array} \right\} \text{Set of constraints}$$

$$x_1, x_2 \geq 0 \quad \left. \begin{array}{l} \text{non-negative constraint} \end{array} \right\}$$

Ques 4

A toy company manufactures two types of doll, a basic version - doll A and a deluxe version - doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2000 per day. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of Rs 3.00 and Rs 5.00 per doll, respectively on doll A and B, then how many of each doll should be produced per day in order to maximize the total profit. Formulate this problem.

Solⁿ: Let x_1 and x_2 be the numbers of dolls produced per day of type A and B, respectively.

Let the doll A require t hrs. so that the doll B requires $2t$ hrs. So the total time to manufacture x_1 and x_2 dolls should not exceed $2000t$ hrs.

$$\therefore t x_1 + 2t x_2 \leq 2000t$$

$$\Rightarrow \text{Max } P = 3x_1 + 5x_2 \quad (\text{objective function})$$

s.t

$$x_1 + 2x_2 \leq 2000$$

$$x_1 + x_2 \leq 1500$$

$$x_2 \leq 600$$

$$x_1 \geq 0, x_2 \geq 0$$

(time constraint)

(Plastic constraint)

(dress constraint)

(non-negativity constraint)

Ques 5

A firm manufactures 3 products A, B and C. The profits are Rs 3, Rs 2 and Rs 4 respectively. The firm has 2 machines and below is the required processing time in minutes for each machine on each product.

		Product		
		A	B	C
Machine	G	4	3	5
	H	2	2	4

Machine G and H have 2,000 and 2,500 machine-minutes respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but no more than 150 A's.

Setup an L.P. problem to maximize profit. Do not solve it.

Solⁿ: Let x_1, x_2, x_3 be the no. of products A, B, C resp.

	A(x_1)	B(x_2)	C(x_3)	Machine minutes
G	4	3	5	2000
H	2	2	4	2500
max. manufacture	100	200	50	

$$\text{Max } P = 3x_1 + 2x_2 + 4x_3 \quad [\text{objective fun}]$$

$$\text{s.t.} \quad 4x_1 + 3x_2 + 5x_3 \leq 2000$$

$$2x_1 + 2x_2 + 4x_3 \leq 2500$$

$$100 \leq x_1 \leq 150$$

$$200 \leq x_2, 50 \leq x_3$$

$$x_1, x_2 \geq 0$$

set of constraints

[non-negativity constraints]

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Graphical Method

It is used only for two variable LPP.

Step 1 Consider each inequality constraint as an equation.

Step 2 Plot each equation on the graph.

Step 3 Shade the common region or feasible region. Every point on this line will satisfy the equation of the line. If the inequality constraint corresponding to that line is less than equal to ' $\leq$ ' then the region below the line and line in the first quadrant due to non-negativity of the variables is shaded.

The points lying in the common region will satisfy all the constraints simultaneously.

Step 4 Find the value of z (objective function) at each corner point of the feasible region.

Step 5 Find out the value of z max/min as per the problem.

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Ques

Solve the LPP using the graphical method

$$\text{Max } z = 5x_1 + 3x_2$$

$$\text{s.t. } 3x_1 + 5x_2 \leq 15$$

$$5x_1 + 2x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

Sol: Step 1: Consider the inequality as eqⁿ

$$3x_1 + 5x_2 = 15 \quad \text{--- (1)}$$

$$5x_1 + 2x_2 = 10 \quad \text{--- (2)}$$

Step 2: Plot the above eqⁿ on the graph

$$3x_1 + 5x_2 = 15$$

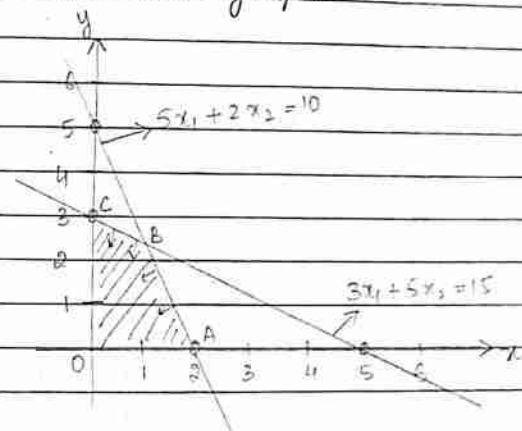
$$x_1 \quad 0 \quad 5 \quad (0, 3)$$

$$x_2 \quad 3 \quad 0 \quad (5, 0)$$

$$5x_1 + 2x_2 = 10$$

$$x_1 \quad 0 \quad 2 \quad (0, 5)$$

$$x_2 \quad 5 \quad 0 \quad (2, 0)$$



Step 3 Shade the common region.
OABC is the feasible region.

Step 4 Find the value of z at each point of the feasible region.

$$\text{Max } z = 5x_1 + 3x_2$$

$$(0, 0) \quad z_0 = 5(0) + 3(0) = 0$$

$$(2, 0) \quad z_A = 5(2) + 3(0) = 10$$

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$$(1.05, 2.33) \quad Z_B = 5(1.05) + 3(2.33) \\ = 5.25 + 7.08 = 12.33$$

$$(0, 3) \quad Z_C = 5(0) + 3(3) = 9$$

$$5x_1 + 2x_2 = 10$$

$$x_2 = 5 - \frac{5}{2}x_1$$

$$10x_1 = 10$$

$$x_1 = \frac{20}{19} = 1.05$$

$$3x_1 + 5x_2 = 15$$

$$3x_1 + 5\left(5 - \frac{5}{2}x_1\right) = 15$$

$$8x_1 + 25 - \frac{25}{2}x_1 = 15$$

$$(6 - \frac{25}{2})x_1 = 15 - 25$$

$$x_2 = 2.33$$

Step 5: Find the max/min value of z .

Max. value of $z = 12.33$ at B.

Min. value of $z = 0$ at O.

$$\therefore \text{max. } z = \frac{235}{19}, \quad x_1 = \frac{20}{19}, \quad x_2 = \frac{45}{19}$$

Ques 2 Solve the LPP using Graphical Method.

$$\text{Max } z = 2x_1 + 3x_2$$

$$\text{s.t.}, \quad x_1 + x_2 \geq 1$$

$$5x_1 - x_2 \geq 0$$

$$x_1 + x_2 \leq 6$$

$$x_1 - 5x_2 \leq 0$$

$$x_2 - x_1 \geq -1$$

$$x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

Ans: Step 1: Consider the inequality as eqⁿ i.e.,

$$x_1 + x_2 = 1 \quad \text{--- (1)}$$

$$5x_1 - x_2 = 0 \quad \text{--- (2)}$$

$$x_1 + x_2 = 6 \quad \text{--- (3)}$$

$$x_1 - 5x_2 = 0 \quad \text{--- (4)}$$

$$x_2 - x_1 = -1 \quad \text{--- (5)}$$

$$x_2 = 3 \quad \text{--- (6)}$$

Step 2 Plot the above equations on a graph.

$$x_1 + x_2 = 1$$

$$x_1 \quad 0 \quad 1 \quad (0, 1)$$

$$x_2 \quad 1 \quad 0 \quad (1, 0)$$

$$5x_1 - x_2 = 0$$

$$x_1 \quad 0 \quad 1 \quad (0, 0)$$

$$x_2 \quad 0 \quad 5 \quad (1, 5)$$

$$x_1 + x_2 = 6$$

$$x_1 \quad 0 \quad 6 \quad (0, 6)$$

$$x_2 \quad 6 \quad 0 \quad (6, 0)$$

$$x_1 - 5x_2 = 0$$

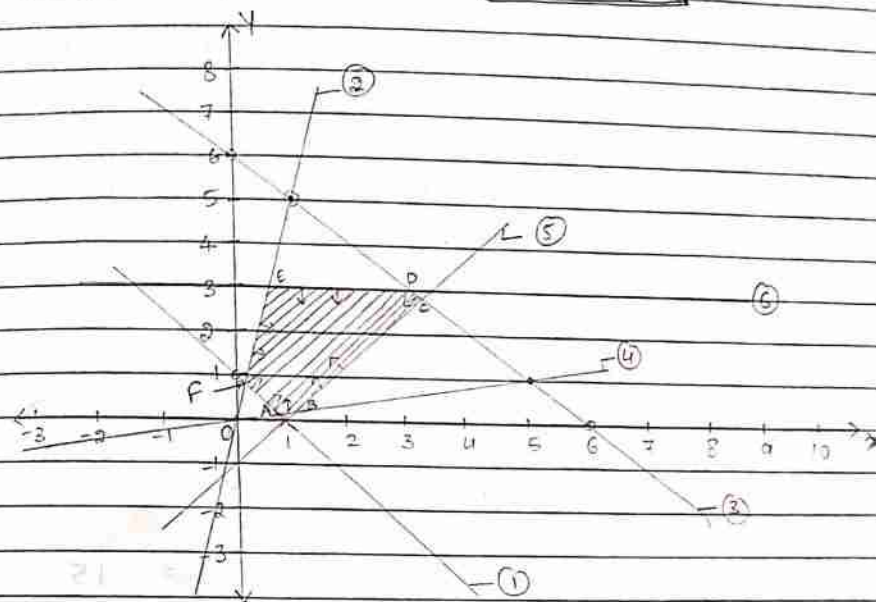
$$x_1 \quad 0 \quad 5 \quad (0, 0)$$

$$x_2 \quad 0 \quad 1 \quad (5, 1)$$

$$x_2 - x_1 = -1$$

$$x_1 \quad 0 \quad 1 \quad (0, -1)$$

$$x_2 \quad -1 \quad 0 \quad (1, 0)$$



Step 3: Shade the common region. ABCDEF is the feasible region.

$$A \rightarrow \text{(1) \& (4)} \quad x_1 + x_2 = 1 \quad x_1 - 5x_2 = 0 \quad A\left(\frac{5}{6}, \frac{1}{6}\right)$$

$$B \rightarrow \text{(4) \& (5)} \quad x_1 - 5x_2 = 0 \quad x_2 - x_1 = -1 \quad B\left(\frac{5}{4}, \frac{1}{4}\right)$$

C → ③ & ⑤

$$\begin{aligned} x_1 + x_2 &= 6 \\ x_1 - x_2 &= 1 \end{aligned} \quad C \left(\frac{7}{2}, \frac{5}{2} \right)$$

D → ③ & ⑥

$$\begin{aligned} x_1 + x_2 &= 6 \\ x_2 &= 3 \end{aligned} \quad D (3, 3)$$

E → ② & ⑥

$$\begin{aligned} 5x_1 - x_2 &= 0 \\ x_2 &= 3 \end{aligned} \quad E \left(\frac{3}{5}, 3 \right)$$

F → ① & ②

$$\begin{aligned} x_1 + x_2 &= 1 \\ 5x_1 - x_2 &= 0 \end{aligned} \quad F \left(\frac{1}{6}, \frac{5}{6} \right)$$

Step 4 Find the value of z at each point of the feasible region
 $\max z = 2x_1 + 3x_2$

$$\left(\frac{5}{2}, \frac{1}{2} \right) z_A = 2\left(\frac{5}{2}\right) + 3\left(\frac{1}{2}\right) = \frac{13}{2} = 2.16$$

$$\left(\frac{5}{4}, \frac{1}{4} \right) z_B = 2\left(\frac{5}{4}\right) + 3\left(\frac{1}{4}\right) = \frac{13}{4} = 3.25$$

$$\left(\frac{7}{2}, \frac{5}{2} \right) z_C = 2\left(\frac{7}{2}\right) + 3\left(\frac{5}{2}\right) = \frac{29}{2} = 14.5$$

$$(3, 3) z_D = 2(3) + 3(3) = 15 = 15$$

$$\left(\frac{3}{5}, 3 \right) z_E = 2\left(\frac{3}{5}\right) + 3(3) = \frac{51}{5} = 10.2$$

$$\left(\frac{1}{6}, \frac{5}{6} \right) z_F = 2\left(\frac{1}{6}\right) + 3\left(\frac{5}{6}\right) = \frac{17}{6} = 2.83$$

Step 5 Find the max/min value of z

Max value of $z = 15$ at point D (3, 3)
 $x_1 = 3$
 $x_2 = 3$

(Minimization Problem)

Ques 3 Solve the LPP using Graphical Method.
 $\min z = 1.5x_1 + 2.5x_2$

$$\begin{aligned} \text{s.t. } x_1 + 3x_2 &\geq 3 \\ x_1 + x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solⁿ Step 1: Consider the inequality of eqⁿ:
 $x_1 + 3x_2 = 3$ — (1)
 $x_1 + x_2 = 2$ — (2)

Step 2: Plot the above equations on a graph.

$$x_1 + 3x_2 = 3$$

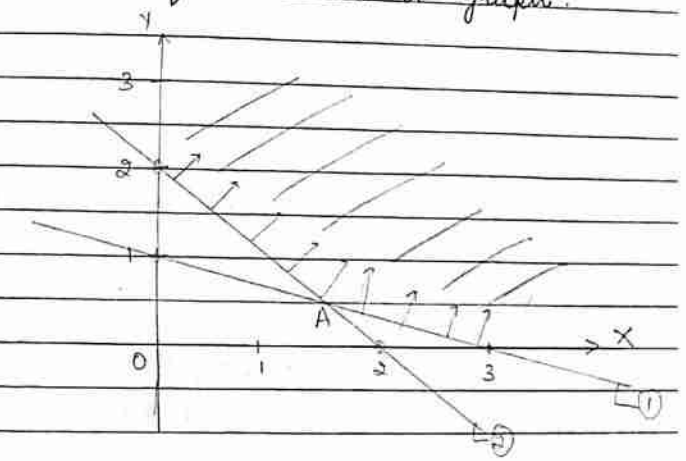
x_1	0	3
x_2	1	0

(0, 1) (3, 0)

$$x_1 + x_2 = 2$$

x_1	0	2
x_2	2	0

(0, 2) (2, 0)



Step 3 Shade the common region

$$\begin{aligned} A \rightarrow x_1 + 3x_2 &= 3 \\ x_1 + x_2 &= 2 \\ \hline 2x_2 &= 1 \\ x_2 &= \frac{1}{2} \\ x_1 &= 2 - \frac{1}{2} = \frac{3}{2} \end{aligned} \quad A \left(\frac{3}{2}, \frac{1}{2} \right)$$

Step 4 Find the value of z at each point of feasible region.

$$z_A = 1.5 \left(\frac{3}{2} \right) + 2.5 \left(\frac{1}{2} \right)$$

$$= \left(\frac{3}{2} \times \frac{3}{2} \right) + \left(\frac{5}{2} \times \frac{1}{2} \right) = \frac{9}{4} + \frac{5}{4} = \frac{14}{4} = 3.5$$

$$z_A = 3.5$$

Step 5 The minimum value is attained at the point of intersection A

$$\min z = 3.5, \quad x_1 = \frac{3}{2}, \quad x_2 = \frac{1}{2}$$

(Problems having unbounded solution)

Ques 4 Solve the LPP using Graphical Method.

$$\text{Max } z = 3x_1 + 2x_2$$

$$\text{s.t. } x_1 - x_2 \leq 1$$

$$x_1 + x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

Solⁿ: Consider the inequality of eqⁿ.

$$x_1 - x_2 = 1 \quad \text{--- (1)}$$

$$x_1 + x_2 = 3 \quad \text{--- (2)}$$

$$x_1 - x_2 = 1$$

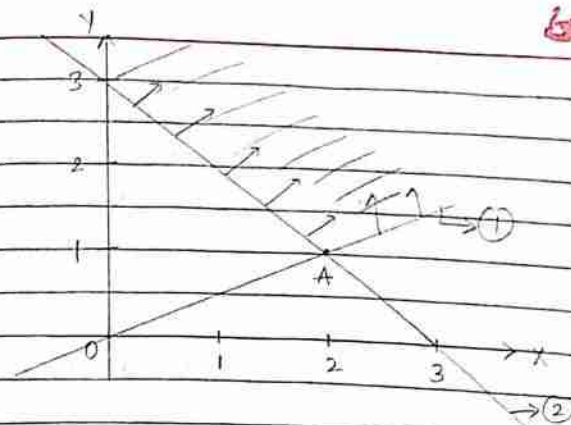
x_1	0	2
x_2	0	1

(1, 0) (2, 1)

$$x_1 + x_2 = 3$$

x_1	0	3
x_2	3	0

(0, 3) (3, 0)



Here z can be made arbitrarily large and problem has no finite max. of z . Such problems are said to have unbounded solution.

(Problems with inconsistent system of constraints)

Ques 5

Solve the LPP using Graphical Method.

$$\text{Max } z = 3x_1 - 2x_2$$

$$\text{s.t. } x_1 + x_2 \leq 1$$

$$2x_1 + 2x_2 \geq 4$$

$$x_1, x_2 \geq 0$$

Solⁿ:

Consider the inequality of eqⁿ

$$2x_1 + 2x_2 = 4 \quad \text{--- (2)}$$

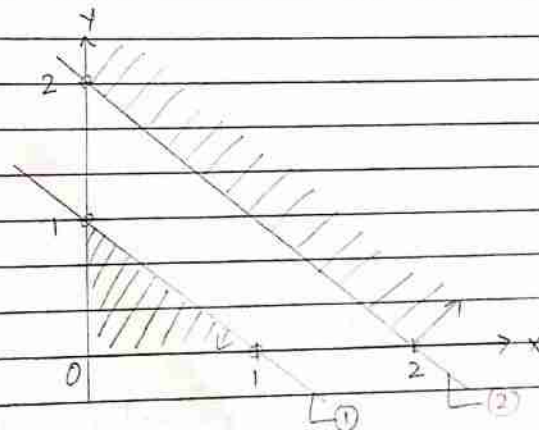
$$x_1 + x_2 = 1 \quad \text{--- (1)}$$

$$x_1 + x_2 = 1$$

x_1	1	0
x_2	0	1

$$2x_1 + 2x_2$$

x_1	0	2
x_2	2	0



Here, there is no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence, the problem has no solution because the constraints are inconsistent.

Ques 6 (Problems in which constraints are equations rather than inequalities.)

Solve the LPP using Graphical Method.

$$\begin{aligned} \text{Max } Z &= 5x_1 + 3x_2 \\ \text{s.t. } 3x_1 + 5x_2 &= 15 \quad \text{--- (1)} \\ 5x_1 + 2x_2 &= 10 \quad \text{--- (2)} \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solⁿ:

$$3x_1 + 5x_2 = 15$$

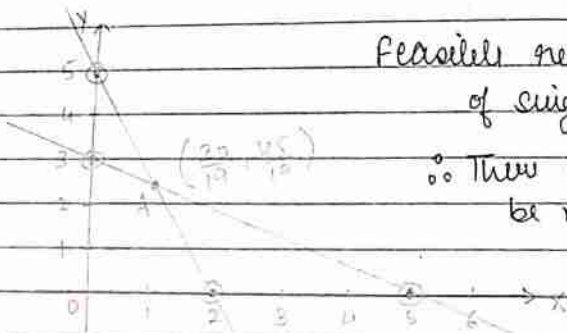
x_1	0	5
x_2	3	0

$(0, 3), (5, 0)$

$$5x_1 + 2x_2 = 10$$

x_1	0	2
x_2	5	0

$(0, 5), (2, 0)$



Feasible region consists of single point A.

$\therefore$ There is nothing to be maximized.

$$\begin{aligned} 3x_1 + 5x_2 &= 15 \quad \text{--- (1)} \\ 5x_1 + 2x_2 &= 10 \quad \text{--- (2)} \end{aligned}$$

$$3\left(\frac{20}{19}\right) - 15 = -5x_2$$

$$5x_2 = 15 - \frac{60}{19}$$

$$x_2 = 3 - \frac{12}{19}$$

$$19x_1 = 20 \Rightarrow x_1 = \frac{20}{19}$$

$$x_2 = \frac{45}{19}$$

Iso-Profit or Iso-Cost Method

In this method, all the previous methods steps are used, then plot the line $z=0$ & the draw lines parallel to $z=0$.

Ques Find the solution of LPP:

$$\begin{aligned} \text{Max } Z &= 45x_1 + 80x_2 \\ \text{s.t. } 5x_1 + 20x_2 &\leq 400 \\ 10x_1 + 15x_2 &\leq 450 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solⁿ:

Consider the inequality of eqⁿ.

$$5x_1 + 20x_2 = 400 \quad \text{--- (1)}$$

$$10x_1 + 15x_2 = 450 \quad \text{--- (2)}$$

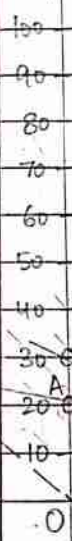
$$5x_1 + 20x_2 = 400$$

x_1	0	80
x_2	20	0

$$10x_1 + 15x_2 = 450$$

x_1	0	45
x_2	30	0

Y



B is intersecting point of (1) & (2).

So, solving these, we get $x_1 = 24, x_2 = 14$

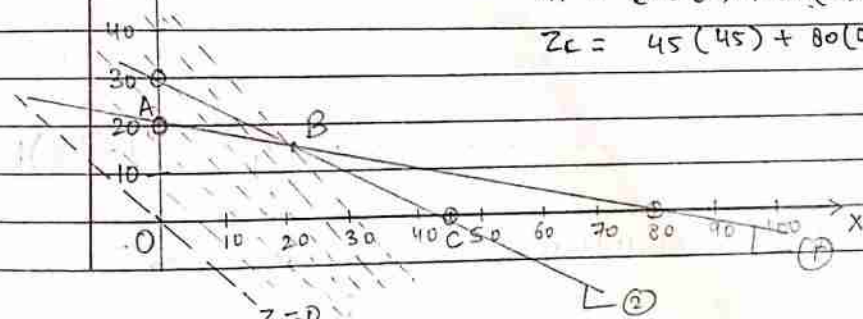
$$\therefore B(24, 14)$$

$$\text{Max } Z_B = 45(24) + 80(14) = 2200$$

$$Z_0 = 45(0) + 80(0) = 0$$

$$Z_A = 45(0) + 80(20) = 1600$$

$$Z_C = 45(45) + 80(0) = 2025$$



Here, there is no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence the problem has no solution because the constraints are inconsistent.

After plotting the lines to $z=0$, B is the last point which lies on the line. $\therefore$ B is the max. point.

Similarly, if we have to get min. point, then it is the first point which lies on the line drawn to $z=0$.

General Linear Programming Problem

The general LPP having 'n' decision variables $x_1, x_2, \dots, x_n$ to maximize/minimize the objective function

$$Z = C_1 x_1 + C_2 x_2 + C_3 x_3 + \dots + C_n x_n$$

Subject to m-constraints:

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n (\leq / = / \geq) b_1$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n (\leq / = / \geq) b_2$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n (\leq / = / \geq) b_m$$

& non-neg restrictions, $x_1, x_2, \dots, x_n \geq 0$

Slack Variables: If a constraint has $\leq$ sign, then in order to make it an equality, we have to add something positive to the left hand side.

The non-negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable.

Eg: Consider the constraints:

$$x_1 + x_2 \leq 2$$

$$2x_1 + 4x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

We add the slack variables $x_3 \geq 0, x_4 \geq 0$ to the left hand sides of the above equations respectively to obtain,

$$x_1 + x_2 + x_3 = 2$$

$$2x_1 + 4x_2 + x_4 = 5$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Surplus Variables: If a constraint has $\geq$ sign, then in order to make it an equality, we have to subtract something non-negative from its LHS.

Thus the positive variable which is subtracted from the LHS of the constraint to convert it into equation is called the surplus variable.

Eg: Consider the constraints:

$$\begin{aligned} x_1 + x_2 &\geq 2 \\ 2x_1 + 4x_2 &\geq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

We subtract the surplus variables $x_3 \geq 0, x_4 \geq 0$ from the left hand side of above inequalities, to obtain

$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 - x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The canonical form: The general formulation of LPP can always be expressed in the form:

$$\begin{aligned} \text{Max } Z &= C_1 x_1 + C_2 x_2 + \dots + C_n x_n \\ \text{s.t. } C & \\ a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n &\leq b_i ; i=1, 2, \dots, m \\ \& \quad x_1, x_2, \dots, x_n &\geq 0 \end{aligned}$$

by making use of some elementary transformations. This form of LPP is called the canonical form of LPP.

The Standard Form:

tp1: All the constraints should be converted into equations except for the non-negativity restrictions which remain as inequalities (≥ 0).

2: The RHS element of each constraint should be non-negative.
 Eg: $3x_1 - 4x_2 \geq -4 \Rightarrow 3x_1 - 4x_2 - x_3 = -4 \Rightarrow -3x_1 + 4x_2 + x_3 = 4$

be made non-negative (if not).

Step3: All variables must have non-negative values.

Step4: The objective function should be of maximized form.

$$\begin{aligned} \text{Min } f(x) &= -\text{Max} [-f(x)] \\ \text{Min } Z &= C_1 x_1 + C_2 x_2 + \dots + C_n x_n \\ \text{Max } Z' &= -C_1 x_1 - C_2 x_2 - \dots - C_n x_n \\ \text{with } Z &= -Z' \end{aligned}$$

Some Important Definitions

1. Solution to LPP: Any set $X = \{x_1, x_2, \dots, x_{n+m}\}$ of variables is called a solution to LPP, if it satisfies the set of constraints only.

2. Feasible Solution (FS): Any set $X = \{x_1, x_2, \dots, x_n\}$ of variables is called a feasible solution (or programme) of L.P. problem, if it satisfies the set of constraints and non-negativity restrictions.

3. Basis Solution (BS): A basic solution to the set of constraints is a solution obtained by setting any n variables (among $m+n$ variables) equal to zero and solving for remaining m variables provided the determinant of the coefficient of these m variables is non-zero. Such m variables are called basic variables.

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and remaining n zero-valued variables are called non-basic variables.

The no. of basic solutions thus obtained will be at the most

$${}^{m+n}C_m = \frac{(m+n)!}{n! m!}$$

which is the no. of combinations of $n+m$ things taken m at a time.

4. Basic Feasible Solution (BFS): A basic feasible solution is a basic solution which also satisfies the non-negativity restrictions.

Basic feasible solutions are of two types in nature:

(i) Non-degenerate BFS:- A non-degenerate basic feasible solution is the basic feasible solution which has exactly m positive x_i ($i=1, 2, \dots, m$).

OR

All m basic variables are positive and the remaining n variables will be zero.

(ii) Degenerate BFS:- A basic feasible solution is called degenerate, if one or more basic variables are zero valued.

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5. Optimum Basic Feasible Solution: A BFS solution is said to be optimum, if it also optimizes (maximizes or minimizes) the objective function.

6. Unbounded Solution: If the value of the objective function z can be increased or decreased indefinitely, such solutions are called unbounded solutions.

Ques Convert the given LPP into standard form

$$\text{Min } Z = 2x_1 - 3x_2$$

$$\text{s.t. } 2x_1 + x_2 \geq -4$$

$$x_1 + x_2 \leq 7$$

$$x_1, x_2 \geq 0$$

Sol: To convert the given LPP into standard form

Use the slack variables.

$$\text{Max } Z' = -2x_1 + 3x_2 \quad \text{where } Z' = -Z$$

$$\text{s.t. } -2x_1 - x_2 \leq 4$$

$$x_1 + x_2 \leq 7$$

$$-2x_1 - x_2 + s_1 = 4$$

$$x_1 + x_2 + s_2 = 7$$

$$x_1, x_2, s_1, s_2 \geq 0$$

4/ Aug/23

SIMPLEX METHOD

Simplex method examines corner points of the feasible region, using matrix row operations, until an optimal solution is found.

Slack variable represents an unused quantity of resource; it is added to less-than or equal-to type constraints in order to get an equality constraints.

Surplus variable represents the amount of resource usage above the minimum required and is added to greater-than or equal-to constraints in order to get equality constraint.

Basic feasible solution: A basic solution to the system of simultaneous equations is called basic feasible solution if $x_B \geq 0$.

C_B : coefficient of basic variable in objective function.

C_j : coefficient of variable in objective function or decision variable.

Degeneracy arises when there is a tie in the minimum ratio value that determines the variable to enter into the next solⁿ.

Basis is the set of variables present in the solution, have positive non-zero value; these variables are also called basic variables.

Non-basic variables are those which are not in the 'basis' and have zero-value.

Infeasibility is the solution in which no solution satisfies all constraints of an LP problem.

$\Delta_j = C_j - Z_j$ row values represent net profit or loss resulting from introducing one unit of any variable into the 'basis' or solution mix.

Key column is in which variable has highest the value corresponding Δ_j .

Key row is in which minimum ratio has value.

Key element is the point where key row and key column intersect.

Ques!

Use simplex method to solve the following LP problem.

$$\text{Max } Z = 5x_1 + 3x_2$$

s.t

$$3x_1 + 5x_2 \leq 15$$

$$5x_1 + 2x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

Solⁿ:

Convert the given LPP into the standard form using slack variables s_1 & s_2 .

$$\text{Max } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2$$

s.t

$$3x_1 + 5x_2 + s_1 = 15$$

$$5x_1 + 2x_2 + s_2 = 10$$

$$x_1, x_2, s_1, s_2 \geq 0$$

CB	Basic variable	x_B	x_1	x_2	s_1	s_2	Minimum Ratio
0	s_1	15	3	5	1	0	$\frac{15}{3} = 5$
0	s_2	10	<u>5</u> key	2	0	1	$\frac{10}{5} = 2 \leftarrow$ out
	$Z_j \rightarrow$		0	0	0	0	
	$\Delta_j \rightarrow$		5	3	0	0	

$\uparrow$ incoming vector

0	s_1	9	0	<u>19/5</u> key	1	-3/5	$\frac{45}{19} = 2.37 \leftarrow$ out
5	x_1	2	1	2/5	0	1/5	5
	$Z_j \rightarrow$		5	2	0	1	
	$\Delta_j \rightarrow$		0	1	0	-1	

$\uparrow$ IN

3	x_2	45/19	0	1	5/19	-3/19	
5	x_1	20/19	1	0	-2/19	5/19	
	$Z_j \rightarrow$		5	3	5/19	16/19	
	$\Delta_j \rightarrow$		0	0	-5/19	-16/19	

Now, all $\Delta_j \leq 0$.

So, $x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$

$\therefore \text{Max } z = 5 \left(\frac{20}{19} \right) + 3 \left(\frac{45}{19} \right)$
 $= \frac{100}{19} + \frac{135}{19} = \frac{235}{19}$

Q2 Solve the following LP using Simplex method:

Min $z = x_1 - 3x_2 + 2x_3$

s.t. $3x_1 - x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Solⁿ: Convert the given LPP into standard form using the slack variables s_1, s_2, s_3 .

Max $z' = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$

s.t.

$3x_1 - x_2 + 2x_3 + s_1 = 7$

$-2x_1 + 4x_2 + s_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

CB	Basic variable	x_B	x_1	x_2	x_3	s_1	s_2	s_3	Mini. Ratio
0	s_1	7	3	-1	2	1	0	0	$\frac{7}{3} = -$
0	s_2	12	-2	<u>4</u> key	0	0	1	0	$\frac{12}{4} = 3 \leftarrow$ out
0	s_3	10	-4	3	8	0	0	1	$\frac{10}{3} = 3.3$
	$Z_j \rightarrow$		0	0	0	0	0	0	
	$\Delta_j \rightarrow$		-1	3	-2	0	0	0	

$\uparrow$ IN

0	s_1	4	<u>5/2</u> key	0	3	1	1/4	0	$\leftarrow$
3	x_2	3	-1/2	1	0	0	1/4	0	-
0	s_3	1	-5/2	0	8	0	-3/4	1	-
	$Z_j \rightarrow$		-3/2	3	0	0	3/4	0	-
	$\Delta_j \rightarrow$		1/2	0	-2	0	-3/4	0	

$\uparrow$

-1	x_1	4	1	0	6/5	2/5	1/10	0	
3	x_2	5	0	1	3/5	1/5	3/10	0	
0	s_3	11	0	0	11	1	-1/2	1	
	$Z_j \rightarrow$		-1	3	3/5	1/5	4/5	0	
	$\Delta_j \rightarrow$		0	0	-13/5	-1/5	-4/5	0	

Since, all the values of $\Delta_j \leq 0$.

So, we have reached the optimal solution.

$x_1 = 4$

Max $z' = 11$

$x_2 = 5$

Min $z = -11$

$x_3 = 0$

GRAPHICAL METHOD (used only for 2-Variable LPP.

Step 1 → Consider each inequality-constraint as an eqⁿ.

2 → Plot each eqⁿ on the graph.

3 → Shade the common/feasible region. Every point on the line will satisfy the eqⁿ of the line.

If the inequality-constraint corresponding to that line is ' $\leq$ ', the region below the line lying in 1st quadrant is shaded.

For the inequality-constraint with ' $\geq$ ' sign, the region above the line in 1st quadrant is shaded.

The points lying in the common region will satisfy all the constraints simultaneously.

4 → find the value of Z (objective funⁿ) at each corner point of the feasible region.

5 → find out the value of Z (max/min) as per the problem.

Example :- (1) find a geometrical interpretation & solⁿ as well for the following LPP :-

$$\text{Max } Z = 3x_1 + 5x_2 \quad \text{s.t.} \quad x_1 + 2x_2 \leq 2000, \\ x_1 + x_2 \leq 1500, \quad x_2 \leq 600 \\ \& \quad x_1 \geq 0, \quad x_2 \geq 0$$

solⁿ Step 1 Consider the inequality as eqⁿ

$$x_1 + 2x_2 = 2000, \quad x_1 + x_2 = 1500, \quad x_2 = 600$$

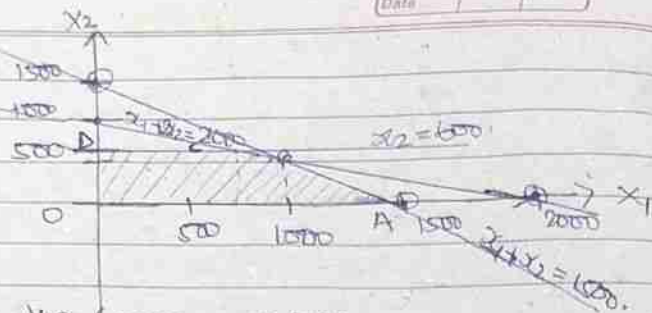
Step 2 → plot the above eqⁿs on graph.

$$x_1 + 2x_2 = 2000$$

x_1	0	2000
x_2	1000	0

$$x_1 + x_2 = 1500$$

x_1	0	1500
x_2	1500	0



Step 3 → shade the common region.

OABCD is the common region.

Step 4 → $Z_0 = 3(0) + 5(0) = 0$

$Z_A = 3(1000) + 5(0) = 4500$

$Z_B = 3(1000) + 5(500) = 5500 \rightarrow \text{MAX}$

$Z_C = 3(0) + 5(600) = 4500$

$Z_D = 3(0) + 5(800) = 2500$

Step 5 → $x_1 = 1000, x_2 = 500$ gives required sol.

2) Max $Z = 8000x_1 + 7000x_2$
s.t. $3x_1 + x_2 \leq 66, x_1 + x_2 \leq 45, x_1 \leq 20, x_2 \leq 40$
& $x_1, x_2 \geq 0$

Solⁿ $3x_1 + x_2 = 66$ — (I)

$x_1 + x_2 = 45$ — (II)

x_1	0	22
x_2	66	0

x_1	0	45
x_2	45	0

$Z_0 = 0$

$Z_A = 8000$ (A is point of intersection of (I) & (II))

∴ solving them, we get coordinates of A.

$3x_1 + x_2 = 66$

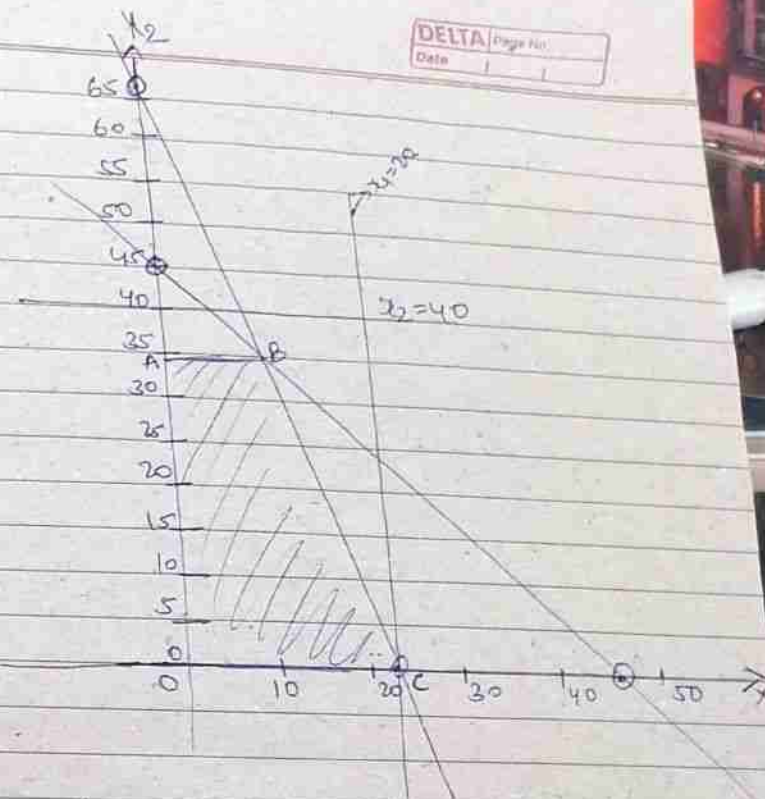
$x_1 + x_2 = 45$

$2x_1 = 21 \Rightarrow x_1 = 10.5, x_2 = 34.5$

$Z_B = 8000(10.5) + 7000(34.5) = 325000 \rightarrow \text{Max}$

$Z_A = 8000(0) + 7000(45) = 241,500$

$Z_C = 8000(20) + 7000(0) = 160000$



3) Minimization Problem

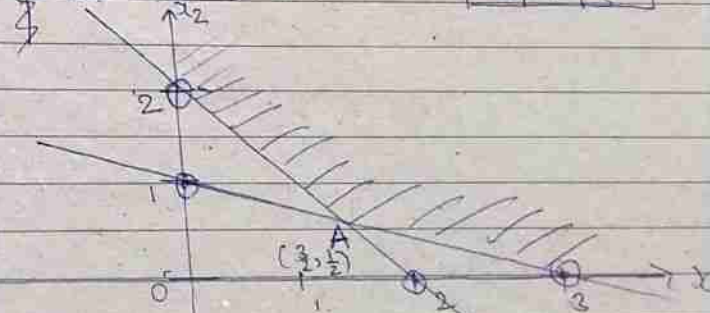
Minimize $Z = 1.5x_1 + 2.5x_2$ s.t. $x_1 + 3x_2 \geq 3, x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0$

Solⁿ $x_1 + 3x_2 = 3$

x_1	0	3
x_2	1	0

x_1	0	2
x_2	2	0



Minimum is attained at $Z_A = 1.5(3) + 2.5(1/2) = 3.5$

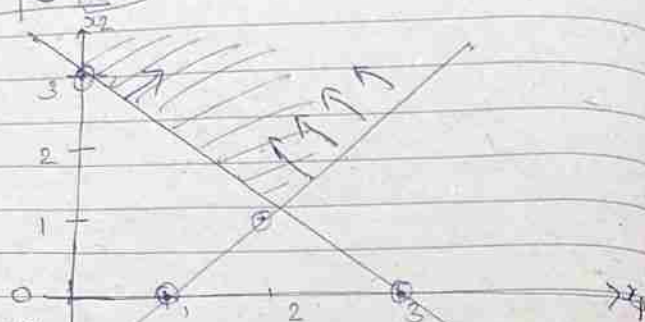
- ④ Problem having Unbounded Solⁿ
Max $Z = 3x_1 + 2x_2$ s.t. $x_1 - x_2 \leq 1$, $x_1 + x_2 \geq 3$ & $x_1, x_2 \geq 0$

Solⁿ $x_1 - x_2 = 1$

x_1	1	2
x_2	0	1

, $x_1 + x_2 = 3$

x_1	0	3
x_2	3	0



Here Z can be made arbitrarily large & problem has no finite max. value of Z . Such problems are said to have unbounded Solⁿ.

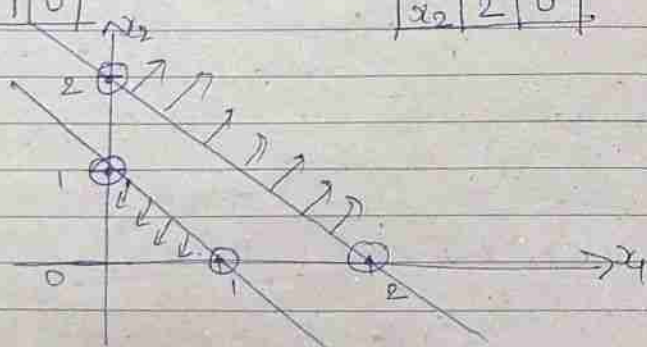
- ⑤ Problem with Inconsistent system of constraints
Maximize $Z = 3x_1 - 2x_2$ s.t. $x_1 + x_2 \leq 1$, $2x_1 + 2x_2 \geq 4$ & $x_1, x_2 \geq 0$

Solⁿ $x_1 + x_2 = 1$

x_1	0	1
x_2	1	0

, $2x_1 + 2x_2 = 4$

x_1	0	2
x_2	2	0



Here there's no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence the problem has no solⁿ because the constraints are inconsistent.

- ⑥ Max $Z = 2x_1 + 3x_2$
s.t. $x_1 + x_2 \geq 1$, $5x_1 - x_2 \geq 0$, $x_1 + x_2 \leq 6$, $x_1 - 5x_2 \leq 7$
 $x_2 - x_1 \geq -1$, $x_2 \leq 3$

Solⁿ $x_1 + x_2 = 1$

x_1	0	1
x_2	1	0

, $5x_1 - x_2 = 0$

x_1	0	1
x_2	0	5

$x_1 + x_2 = 6$

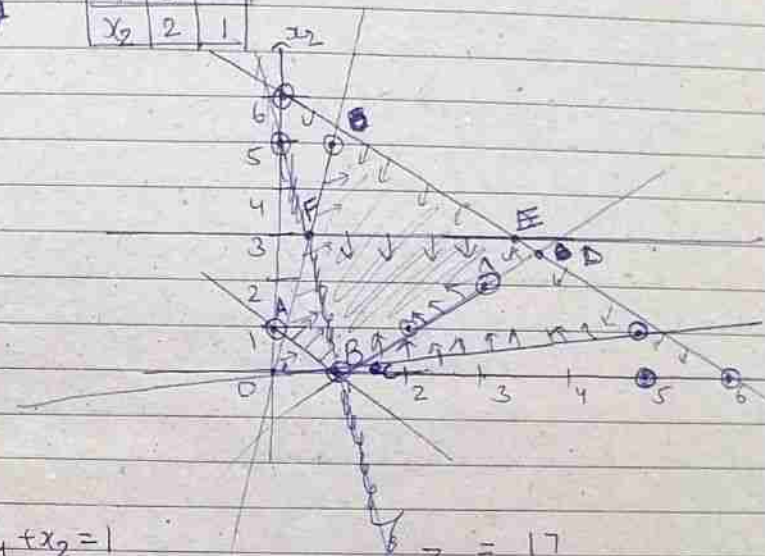
x_1	0	6
x_2	6	0

, $x_1 - 5x_2 = 7$

x_1	5	0
x_2	1	0

$x_2 - x_1 = -1$

x_1	3	2
x_2	2	1



A $\rightarrow x_1 + x_2 = 1$
 $5x_1 - x_2 = 0$
 $6x_1 = 1 \rightarrow x_1 = \frac{1}{6}, x_2 = \frac{5}{6}$

$Z_A = \frac{17}{6}$

B $\rightarrow x_1 - 5x_2 = 7$
 $x_1 + x_2 = 1$
 $x_2 = \frac{1}{6}, x_1 = \frac{5}{6}$

$Z_B = \frac{13}{6}$

C $\rightarrow x_1 - 5x_2 = 0$
 $-x_1 + x_2 = 1$
 $x_2 = \frac{1}{4}, x_1 = \frac{5}{4}$

$Z_C = \frac{13}{4}$

$$\begin{aligned} D \rightarrow x_2 - x_1 &= -1 \\ x_2 + x_1 &= 6 \\ \hline 2x_2 &= 5 \\ x_2 &= \frac{5}{2}, x_1 = \frac{7}{2} \end{aligned}$$

$$Z_D = \frac{29}{2}$$

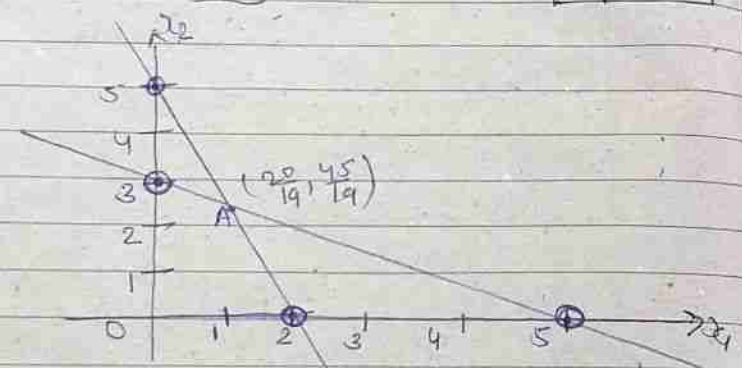
$$\begin{aligned} E \rightarrow (3, 3), Z_E &= 15 \\ F \rightarrow (3, 8), Z_F &= 51 \end{aligned}$$

So, Max at E.

⑦ Problem in which constraints are equations rather than inequalities

$$\text{Max } Z = 5x_1 + 3x_2 \quad \text{s.t.} \quad 3x_1 + 5x_2 = 15, \quad 5x_1 + 2x_2 = 10, \quad x_1, x_2 \geq 0$$

$$\text{eg}^n \quad 3x_1 + 5x_2 = 15 \quad \begin{array}{c|c|c} x_1 & 0 & 3 \\ x_2 & 3 & 0 \end{array}, \quad 5x_1 + 2x_2 = 10 \quad \begin{array}{c|c|c} x_1 & 0 & 2 \\ x_2 & 5 & 0 \end{array}$$



Since, there is only a single solⁿ point $A(\frac{20}{19}, \frac{45}{19})$, there's nothing to be maximized.

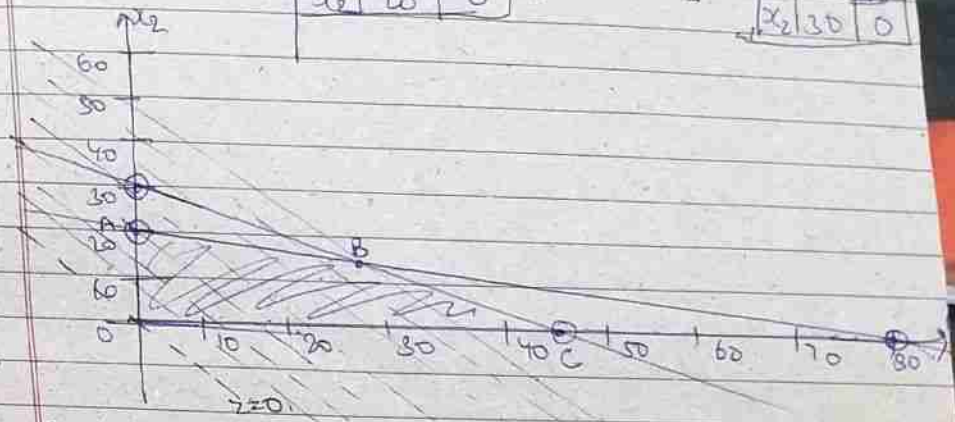
* Iso Profit or Isocost method

In this method all the previous method steps are used, then plot the line $z=0$ & then draw lines $\parallel$ to $z=0$

Q find the solⁿ of LPP

$$\text{Max } z = 45x_1 + 80x_2 \quad \text{s.t.} \quad 5x_1 + 2x_2 \leq 100, \quad 10x_1 + 15x_2 \leq 450, \quad x_1, x_2 \geq 0$$

$$\text{sol}^n: \quad 5x_1 + 2x_2 = 100 \quad \begin{array}{c|c|c} x_1 & 0 & 20 \\ x_2 & 50 & 0 \end{array}, \quad 10x_1 + 15x_2 = 450 \quad \begin{array}{c|c|c} x_1 & 0 & 45 \\ x_2 & 30 & 0 \end{array}$$



B is at intersecting point of eq^s $5x_1 + 2x_2 = 100$ & $10x_1 + 15x_2 = 450$. So, solving these we get $x_1 = 24, x_2 = 14$. $\therefore B(24, 14) \rightarrow Z = 2200 \rightarrow \text{Max}$.
 $O(0, 0) \rightarrow Z = 0$
 $A(0, 20) \rightarrow Z = 1600$
 $C(45, 0) \rightarrow Z = 2025$

⇒ After plotting $\parallel$ lines to $z=0$, B is the last point which lies on that $\parallel$ line. $\therefore$ B is the Max point.
 If we've to get min point then it's the 1st point which lies on the $\parallel$ line drawn to $z=0$

* General LPP

The general LPP having 'n' decision variables $x_1, x_2, \dots, x_n$ to maximize/minimize the objective funⁿ, $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$ Subject to the 'm' constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n (\leq/\geq/=) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n (\leq/\geq/=) b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n (\leq/\geq/=) b_m$$

& non-ve restrictions, $x_1, x_2, \dots, x_n \geq 0$.

* SLACK VARIABLE

If a constraint has $\leq$ sign, then in order to make it an equality, we've to add something +ve to the L.H.S.

→ The non-ve variable which is added to the L.H.S. of the constraint to convert it into eqⁿ is called slack variable.

Ex: $x_1 + x_2 \leq 2$

To convert it into equality we add s_1 to it

i.e. $x_1 + x_2 + s_1 = 2$ → slack variable.

* SURPLUS VARIABLE

If a constraint has $\geq$ sign, then in order to make it an equality we've to subtract something non-ve. from its L.H.S.

→ Thus the +ve variable which is subtracted from L.H.S. of the constraint to convert it into eqⁿ is called surplus variable

Ex: $2x_1 + 3x_2 \geq 7$

To convert it into equality we've to subtract s_2

from it.

i.e. $x_1 + x_2 - s_2 = 7$ → surplus variable.

Std. form of LPP

step 1 All the constraints should be converted into the eq^s except for the non-negativity restrictions which remain as inequalities (≥ 0).

step 2 The R.H.S. of each constraint should be made non-ve (if not)

step 3 All variables must have non-ve values.

A variable which is unrestricted in sign is equivalent to the difference betⁿ two non-ve variables.

step 4 The objective funⁿ should be of maximization form.

Some Imp def^s

① Solⁿ to LPP: Any set $x = \{x_1, x_2, \dots, x_n\}$ is called solⁿ of LPP if it satisfies the set of constraints.

② Feasible Solⁿ → Any set $x = \{x_1, x_2, \dots, x_n\}$ of variables is called feasible solⁿ of LPP if it satisfies the set of constraints & non-ve restrictions also.

③ Basic Solⁿ → A basic solⁿ to the set of constraints is a solⁿ obtained by setting any 'n' variables among (n+m) variables equal to zero & solving for the remaining 'm' variables provided the determinant of the coefficients of these 'm'

Variables & non-zero.
 Such 'm' variables are called Basic Variables
 & remaining 'n' zero valued variables are
 called non-basic variables.

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Basic feasible solⁿ

A basic feasible solⁿ is a basic solⁿ which
 also satisfies the non-negativity restrictions.

→ These are of two types.

a) Non-degenerate BFS-

All the m basic variables are +ve & remaining
 'n' variables will be zero.

b) Degenerate BFS-

A BFS is called degenerate, if one or more
 basic variables are zero-valued.

5* Optimum BFS

A BFS is s.t.b. optimum, if it also optimizes
 (maximizes or minimizes) the objective funⁿ.

6) Unbounded Solⁿ

If the value of the objective funⁿ Z can be
 increased or decreased indefinitely, such solⁿs are
 called unbounded solⁿs.

Q Convert the given LPP into std. form

$$\text{Min } Z = 2x_1 - 3x_2$$

$$\text{s.t. } 2x_1 + x_2 \geq -4$$

$$x_1 + x_2 \leq 7$$

$$x_1, x_2 \geq 0$$

Solⁿ To convert the given LPP into std form

use the slack variables

$$\text{Max } Z' = -2x_1 - 3x_2 \quad \text{Where } Z' = -Z$$

$$\text{s.t. } -2x_1 - x_2 \leq 4$$

$$x_1 + x_2 \leq 7$$

$$-2x_1 - x_2 + s_1 = 4$$

$$x_1 + x_2 + s_2 = 7$$

$$x_1, x_2, s_1, s_2 \geq 0$$

SIMPLEX METHOD

$$\text{Max } Z = 5x_1 + 3x_2$$

$$\text{s.t. } 3x_1 + 5x_2 \leq 15$$

$$5x_1 + 2x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

Solⁿ Convert the given LPP into the std. form using
 slack variables s_1 & s_2

$$\text{Max } Z = 5x_1 + 3x_2 + 0s_1 + 0s_2$$

$$\text{s.t. } 3x_1 + 5x_2 + s_1 = 15$$

$$\text{As an decision variable } 5x_1 + 2x_2 + s_2 = 10$$

$$x_1, x_2, s_1, s_2 \geq 0$$

C_B : Coeff. of basic variable in objective funⁿ

X_B : Basic feasible solⁿ

C_j : Coeff. of variables in objective funⁿ

Δ_j : Net evaluation which is computed using formula

$$\Delta_j = Z_j - C_j$$

$$Z = C_B X_B$$

$$Z_j = C_j X_j$$

X_k : Key Column → wo column jisme Δ_j ke corresponding
 highest +ve. variable hoga.

Key Row → wo row jo Min. ratio column me min.
 value degi.

Key element → the point where key column & key row
 intersect.

Basic variable two variable note h jo matrix me unit matrix bnate h.

		$q \rightarrow 35 \quad 3 \quad 0 \quad 0$					Minimum ratio x_B/x_k for $x_k > 0$
C_B	Basic Variables	x_B	x_1	x_2	s_1	s_2	
0	s_1	15	3	5	1	0	$x_B/x_1 = \frac{15}{3} = 5$
0	s_2	10	5	2	0	1	$x_B/x_1 = \frac{10}{5} = 2 \leftarrow$ outgoing key row is 2nd row
$Z = C_B x_B = 0$		$q \rightarrow 0$	0	0	0	0	
$\Delta_j = C_j - Z_j \rightarrow$		5	3	0	0	0	
incoming vector so key column is x_1							
0	s_1	9	0	$\frac{14}{5}$	1	$\frac{3}{5}$	$x_B/x_2 = \frac{9}{14/5} = \frac{45}{14} \leftarrow$ outgoing
5	x_1	2	1	$\frac{2}{5}$	0	$\frac{1}{5}$	$x_B/x_2 = \frac{2}{2/5} = 5$
$Z = C_B x_B = 10$		$q \rightarrow 5$	2	0	1		$x_B/x_2 =$
$\Delta_j = C_j - Z_j$		0	0	0	-1		
incoming vector so key column is x_2							
3	x_2	$\frac{45}{19}$	0	1	$\frac{5}{19}$	$-\frac{3}{19}$	
5	x_1	$\frac{20}{19}$	1	0	$-\frac{2}{19}$	$\frac{5}{19}$	
$Z_j \rightarrow$		5	3	$\frac{5}{19}$	$\frac{16}{19}$		
$\Delta_j \rightarrow$		0	0	$-\frac{5}{19}$	$-\frac{16}{19}$		

Now all $\Delta_j \leq 0$

So, $x_1 = \frac{20}{19}$, $x_2 = \frac{45}{19}$

Now, $\text{Max } Z = 5\left(\frac{20}{19}\right) + 3\left(\frac{45}{19}\right) = \frac{235}{19}$

Q2 $\text{Max } Z = 5x_1 + 7x_2$ s.t. $x_1 + x_2 \leq 4$, $3x_1 - 2x_2 \leq 24$, $10x_1 + 7x_2 \leq 35$

Solⁿ $\text{Max } Z = 5x_1 + 7x_2 + 0s_1 + 0s_2 + 0s_3$
s.t. $x_1 + x_2 + s_1 = 4$, $3x_1 - 2x_2 + s_2 = 24$, $10x_1 + 7x_2 + s_3 = 35$

		$q \rightarrow 5 \quad 7 \quad 0 \quad 0 \quad 0$						Minimum ratio x_B/x_k
C_B	Basic Variables	x_B	x_1	x_2	s_1	s_2	s_3	
0	s_1	4	1	1	1	0	0	$x_B/x_2 = \frac{4}{1} = 4 \rightarrow$
0	s_2	24	3	-8	0	1	0	$= \frac{24}{-8}$
0	s_3	35	10	7	0	0	1	$= \frac{35}{7} = 5$
$Z_j \rightarrow$		0	0	0	0	0	0	
$\Delta_j \rightarrow$		5	7	0	0	0	0	
incoming vector so key column is x_2								
7	x_2	4	1	1	1	0	0	
0	s_2	56	11	0	8	1	0	
0	s_3	7	3	0	-7	0	1	
$Z_j \rightarrow$		7	7	7	0	0	0	
$\Delta_j \rightarrow$		-2	0	-7	0	0	0	

Since all $\Delta_j \leq 0$

So, $x_1 = 0$,
 $x_2 = 4$

$\text{Max } Z = 28$

Q3 $\text{Max } Z = x_1 + x_2 + 3x_3$
s.t. $3x_1 + 2x_2 + x_3 \leq 3$
 $2x_1 + x_2 + 2x_3 \leq 2$
 $x_1, x_2, x_3 \geq 0$

Solⁿ $\text{Max } Z = x_1 + x_2 + 3x_3 + 0s_1 + 0s_2$
s.t. $3x_1 + 2x_2 + x_3 + s_1 = 3$
 $2x_1 + x_2 + 2x_3 + s_2 = 2$

		Gj → 1 1 3 0 0							Minimum Ratio x_b/x_k
C_b	Basic Variable	x_b	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	3	3	2	1	1	0	0	$x_b/x_3 = \frac{3}{1} = 3$ $= \frac{2}{2} = 1 \rightarrow$
0	s_2	2	2	1	2	0	1	0	
	$Z_j \rightarrow$		0	0	0	0	0	0	
	$\Delta_j \rightarrow$		1	1	3	0	0	0	
					↑				
0	s_1	2	2	$\frac{3}{2}$	0	1	$-\frac{1}{2}$		
3	x_3	1	1	$\frac{1}{2}$	1	0	$\frac{1}{2}$		
	$Z_j \rightarrow$		3	$\frac{3}{2}$	3	0	$\frac{3}{2}$		
	$\Delta_j \rightarrow$		-2	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$		

Since $\Delta_j \leq 0$
hence, $x_1 = 0, x_2 = 0, x_3 = 1$
 $\therefore \text{Max } Z = 3$

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Q Solve the following LPP using Simplex Method.

Min. $Z = x_1 - 3x_2 + 2x_3$

s.t. $3x_1 - x_2 + 3x_3 \leq 7, -2x_1 + 4x_2 \leq 12$

$-4x_1 + 3x_2 + 8x_3 \leq 10$

$x_1, x_2, x_3 \geq 0$

Sol Convert the given LPP into the std. form using slack variables s_1, s_2, s_3

Max $Z' = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$ ($Z' = -Z$)

s.t. $3x_1 - x_2 + 3x_3 + s_1 = 7$

$-2x_1 + 4x_2 + s_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

		Gj → -1 3 -2 0 0 0							Min Ratio
C_b	Basic Variable	x_b	x_1	x_2	x_3	s_1	s_2	s_3	
0	s_1	7	3	-1	3	1	0	0	$x_b/x_2 = \frac{7}{-1} = -7$ $\frac{12}{-2} = -6 \rightarrow$ $\frac{10}{-4} = -2.5$
0	s_2	12	-2	4	0	0	1	0	
0	s_3	10	-4	3	8	0	0	1	
	$Z_j \rightarrow$		0	0	0	0	0	0	
	$\Delta_j \rightarrow$		-1	3	-2	0	0	0	
				↑					
0	s_1	10	$\frac{5}{2}$	0	3	1	$\frac{1}{4}$	0	$x_b/x_1 = \frac{10}{5/2} = 4 \rightarrow$ $3 \div (-1/2) = -6$ $1 \div (-5/2) = -$
3	x_2	3	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0	
0	s_3	1	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1	
	$Z_j \rightarrow$		$-\frac{3}{2}$	3	0	0	$\frac{3}{4}$	0	
	$\Delta_j \rightarrow$		$\frac{1}{2}$	0	-2	0	$-\frac{3}{4}$	0	
			↑						
	x_1	4	1	0	$\frac{6}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0	
3	x_2	5	0	1	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0	
0	s_3	11	0	0	11	1	$-\frac{1}{2}$	1	
	$Z_j \rightarrow$		-1	3	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{8}{10}$	0	
	$\Delta_j \rightarrow$		0	0	$-\frac{11}{5}$	$-\frac{1}{5}$	$-\frac{8}{10}$	0	

$x_1 = 4, x_2 = 5, x_3 = 0$

Max $Z' = -4 + 15 = 11$

Min $Z = -11$

Assignment

Q1) Min $Z = x_1 - 3x_2 + 2x_3$ Max $Z = 3x_1 + 2x_2 + 5x_3$ s.t.
 $s.t. \quad 3x_1 - x_2 + 2x_3 \leq 7 \quad x_1 + 2x_2 + x_3 \leq 430$
 $-2x_1 + 4x_2 \leq 12 \quad 3x_1 + 2x_2 \leq 460, \quad x_1 + 4x_2 \leq 420$
 and $x_1, x_2, x_3 \geq 0$.

Self Std. form $\rightarrow$ Max $Z = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$

s.t. $x_1 + 2x_2 + x_3 + s_1 = 430$

$3x_1 + 2x_2 + s_2 = 460$

$x_1 + 4x_2 + s_3 = 420$

$\rightarrow 3 \quad 2 \quad 5 \quad 0 \quad 0 \quad 0$

C_B	Basic Variables	x_B	x_1	x_2	x_3	s_1	s_2	s_3	Min. Ratio
0	s_1	430	1	2	1	1	0	0	$x_2/x_1 = 430$
0	s_2	460	3	0	2	0	1	0	$= \frac{460}{2} \rightarrow$
0	s_3	420	1	4	0	0	0	1	
	$Z_j \rightarrow$		0	0	0	0	0	0	
	$\Delta_j \rightarrow$		3	2	5	0	0	0	

0	s_1	200	$-\frac{1}{2}$	2	0	1	$-\frac{1}{2}$	0	$x_2/x_2 = 100 \rightarrow$
5	x_2	230	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	
0	s_3	420	1	4	0	0	0	1	105
	$Z_j \rightarrow$		$\frac{15}{2}$	0	5	0	$\frac{5}{2}$	0	
	$\Delta_j \rightarrow$		$-\frac{9}{2}$	2	0	0	$-\frac{5}{2}$	0	

2	x_2	100	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	
5	x_3	230	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	
0	s_3	20	2	0	0	-2	1	1	
	$Z_j \rightarrow$		7	2	5	1	2	0	
	$\Delta_j \rightarrow$		-4	0	0	-1	-2	0	

$s_1, x_1 = 0, x_2 = 100, x_3 = 230$

max $Z = 1350$

Q2) Max $Z = 3x_1 + 5x_2 + 4x_3$ s.t. $2x_1 + 3x_2 \leq 8, 2x_2 + 5x_3 \leq 10,$
 $3x_1 + 2x_2 + 4x_3 \leq 15$ and $x_1, x_2, x_3 \geq 0$

Self Std. form $\rightarrow$ Max $Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$

s.t. $2x_1 + 3x_2 + s_1 = 8$

$2x_2 + 5x_3 + s_2 = 10$

$3x_1 + 2x_2 + 4x_3 + s_3 = 15$

$\rightarrow 3 \quad 5 \quad 4 \quad 0 \quad 0 \quad 0$

C_B	Basic Variable	x_B	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio
0	s_1	8	2	3	0	1	0	0	$x_2/x_2 = \frac{8}{3} \rightarrow$
0	s_2	10	0	2	5	0	1	0	5
0	s_3	15	3	2	4	0	0	1	$\frac{15}{2}$
	$Z_j \rightarrow$		0	0	0	0	0	0	
	$\Delta_j \rightarrow$		3	5	4	0	0	0	

5	x_2	$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$x_2/x_2 =$
0	s_2	$\frac{14}{3}$	$-\frac{4}{3}$	0	5	$-\frac{2}{3}$	1	0	$\frac{14}{3} = 5 = \frac{14}{3} \rightarrow$
0	s_3	$\frac{29}{3}$	$\frac{5}{3}$	0	4	$-\frac{2}{3}$	0	1	$\frac{29}{3} = 4 = \frac{29}{3}$
	$Z_j \rightarrow$		$\frac{10}{3}$	5	0	$\frac{5}{3}$	0	0	
	$\Delta_j \rightarrow$		$\frac{1}{3}$	0	4	$-\frac{5}{3}$	0	0	

5	x_2	$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$x_2/x_1 = 4$
4	x_3	$\frac{14}{15}$	$-\frac{4}{15}$	0	1	$-\frac{2}{15}$	$\frac{1}{15}$	0	
0	s_3	$\frac{89}{15}$	$\frac{41}{15}$	0	0	$-\frac{2}{15}$	$-\frac{4}{15}$	1	$\frac{89}{15} \rightarrow$
	$Z_j \rightarrow$		$\frac{34}{15}$	5	4	$\frac{17}{15}$	$\frac{4}{15}$	0	
	$\Delta_j \rightarrow$		$\frac{11}{15}$	0	0	$-\frac{17}{15}$	$-\frac{14}{15}$	0	

10 - 16	x_2	$\frac{50}{41}$	0	1	0	$\frac{15}{41}$	$\frac{5}{41}$	$\frac{16}{41}$	
30 - 16	x_3	$\frac{62}{41}$	0	0	1	$-\frac{6}{41}$	$\frac{14}{41}$	$\frac{15}{41}$	
89 - 16	x_1	$\frac{89}{41}$	1	0	0	$-\frac{2}{41}$	$\frac{12}{41}$	$\frac{15}{41}$	
	$Z_j \rightarrow$		3	5	4	$\frac{11}{41}$	$\frac{17}{41}$	$\frac{15}{41}$	
	$\Delta_j \rightarrow$		0	0	0	$-\frac{11}{41}$	$-\frac{17}{41}$	$-\frac{15}{41}$	

$x_1 = \frac{89}{41}, x_2 = \frac{50}{41}, x_3 = \frac{62}{41}, \max Z = \frac{765}{41}$

83 Minimize $Z = 8x_1 - 3x_2 + 2x_3$ s.t. $3x_1 - x_2 + 2x_3 \leq 7$,
 $-2x_1 + 4x_2 \leq 12$, $-4x_1 + 3x_2 + 8x_3 \leq 10$ & $x_1, x_2, x_3 \geq 0$

Std-form $\max Z' = -x_1 + 3x_2 - 2x_3 + 0s_1 + 0s_2 + 0s_3$

s.t. $3x_1 - x_2 + 2x_3 + s_1 = 7$

$-2x_1 + 4x_2 + s_2 = 12$

$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$

ys -1 3 -2 0 0 0

CB	Basic Variable	x_0	x_1	x_2	x_3	s_1	s_2	s_3	Min-Ratio
----	----------------	-------	-------	-------	-------	-------	-------	-------	-----------

0	s_1	7	3	-1	2	1	0	0	$x_0/x_1 = -$
---	-------	---	---	----	---	---	---	---	---------------

0	s_2	12	-2	4	0	0	1	0	3 $\rightarrow$
---	-------	----	----	---	---	---	---	---	-----------------

0	s_3	10	-4	3	8	0	0	1	3.233
---	-------	----	----	---	---	---	---	---	-------

$Z_j \rightarrow 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$

$\Delta_j \rightarrow -1 \ 3 \ -2 \ 0 \ 0 \ 0$

$\uparrow$

0	s_1	10	5/2	0	2	1	1/4	0	$x_0/x_1 = 4 \rightarrow$
---	-------	----	-----	---	---	---	-----	---	---------------------------

3	x_2	3	-1/2	1	0	0	1/4	0	-
---	-------	---	------	---	---	---	-----	---	---

0	s_3	1	-5/2	0	8	0	-3/4	1	-
---	-------	---	------	---	---	---	------	---	---

$Z_j \rightarrow -3/2 \ 3 \ 0 \ 0 \ 3/4 \ 0$

$\Delta_j \rightarrow 1/2 \ 0 \ -2 \ 0 \ -3/4 \ 0$

$\uparrow$

-1	x_1	4	1	0	4/5	2/5	1/10	0	
----	-------	---	---	---	-----	-----	------	---	--

3	x_2	5	0	1	2/5	9/5	3/10	0	
---	-------	---	---	---	-----	-----	------	---	--

0	s_3	11	0	0	10	1	-1/5	1	
---	-------	----	---	---	----	---	------	---	--

$Z_j \rightarrow -1 \ 3 \ 2/5 \ 1/5 \ 4/5 \ 0$

$\Delta_j \rightarrow 0 \ 0 \ -12/5 \ -1/5 \ -4/5 \ 0$

$x_1 = 4, x_2 = 5, x_3 = 0$

$\max Z' = 11$

$\min Z = -11$

84 Max $Z = 3x_1 + 2x_2$ s.t. $2x_1 + x_2 \leq 5, x_1 + x_2 \leq 3$,
 $x_1, x_2 \geq 0$

Std-form $\max Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$

s.t. $2x_1 + x_2 + s_1 = 5, x_1 + x_2 + s_2 = 3$

ys 3 2 0 0

CB	Basic Variable	x_0	x_1	x_2	s_1	s_2	Min-Ratio
----	----------------	-------	-------	-------	-------	-------	-----------

0	s_1	5	2	1	1	0	$x_0/x_1 = \frac{5}{2} = 2.5 \rightarrow$
---	-------	---	---	---	---	---	---

0	s_2	3	1	1	0	1	= 3
---	-------	---	---	---	---	---	-----

$Z_j \rightarrow 0 \ 0 \ 0 \ 0$

$\Delta_j \rightarrow 3 \ 2 \ 0 \ 0$

$\uparrow$

3	x_1	5/2	1	1/2	1/2	0	$x_0/x_2 = 5$
---	-------	-----	---	-----	-----	---	---------------

0	s_1	1/2	0	1/2	-1/2	1	= 1 $\rightarrow$
---	-------	-----	---	-----	------	---	-------------------

$Z_j \rightarrow 3 \ 3/2 \ 3/2 \ 0$

$\Delta_j \rightarrow 0 \ 1/2 \ -3/2 \ 0$

$\uparrow$

3	x_1	2	1	0	1	-1	
---	-------	---	---	---	---	----	--

2	x_2	1	0	1	-1	2	
---	-------	---	---	---	----	---	--

$Z_j \rightarrow 3 \ 2 \ 1 \ 1$

$\Delta_j \rightarrow 0 \ 0 \ -1 \ -1$

$\Rightarrow x_1 = 2, x_2 = 1$ so, $\max Z = 8$

85 Max $Z = 5x_1 + 3x_2$ s.t. $x_1 + x_2 \leq 2, 5x_1 + 2x_2 \leq 10$

$3x_1 + x_2 \leq 12, x_1, x_2 \geq 0$

Std-form $\max Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$

s.t. $x_1 + x_2 + s_1 = 2$

$5x_1 + 2x_2 + s_2 = 10$

$3x_1 + x_2 + s_3 = 12$

		g → 5 3 0 0						Min Ratio
C _B	Basic Variable	x _B	x ₁	x ₂	s ₁	s ₂	s ₃	
0	s ₁	2	1	1	1	0	0	x _B /x ₁ = 2 → 2 min 4
0	s ₂	10	5	2	0	1	0	
0	s ₃	12	3	0	0	0	1	
	z _j	0	0	0	0	0	0	
	Δ _j		5	3	0	0	0	
5	x ₁	2	1	1	1	0	0	
0	s ₂	0	0	-3	-5	1	0	
0	s ₃	6	0	5	-3	0	1	
	z _j	5	5	5	0	0	0	
	Δ _j	0	-2	-5	0	0	0	

x₁ = 2, x₂ = 0 s.o., Max Z = 10

Q6 Max Z = 2x₁ + 4x₂ + x₃ + x₄ s.t. 2x₁ + x₂ + 2x₃ + 3x₄ ≤ 12
3x₁ + 2x₂ + 2x₄ ≤ 20, 2x₁ + x₂ + 4x₃ ≤ 16,
x₁, x₂, x₃, x₄ ≥ 0

solⁿ Std. form, Max Z = 2x₁ + 4x₂ + x₃ + x₄ + 0s₁ + 0s₂ + 0s₃

s.t. 2x₁ + x₂ + 2x₃ + 3x₄ + s₁ = 12

3x₁ + 2x₂ + 2x₄ + s₂ = 20

2x₁ + x₂ + 4x₃ + s₃ = 16

g → 2 4 1 1 0 0 0

C _B	Basic Variable	x _B	x ₁	x ₂	x ₃	x ₄	s ₁	s ₂	s ₃	Min Ratio
0	s ₁	12	2	1	2	3	1	0	0	x _B /x ₂ = 12 →
0	s ₂	20	3	0	2	2	0	1	0	
0	s ₃	16	2	1	4	0	0	0	1	16
	z _j	0	0	0	0	0	0	0	0	
	Δ _j	2	4	1	1	0	0	0	0	

4	x_2	12	2	1	2	3	1	0	0
0	s_2	20	3	0	2	2	0	1	0
0	s_3	4	0	0	2	-3	-1	0	0
	$z_j \rightarrow$	8	4	8	12	4	0	0	0
	$\Delta_j \rightarrow$	-6	0	-7	-11	-4	0	0	0

s.o., x₁ = 0, x₂ = 12, x₃ = 0

Max Z = 48

Q7 Max Z = 2x + 5y s.t. x + y ≤ 600; 0 ≤ x ≤ 400,
0 ≤ y ≤ 300

solⁿ Std. form Max Z = 2x + 5y + 0s₁ + 0s₂ + 0s₃

s.t. x + y + s₁ = 600

x + s₂ = 400

y + s₃ = 300

g → 2 5 0 0 0

C _B	Basic Variable	x _B	x	y	s ₁	s ₂	s ₃	Min Ratio
0	s ₁	600	1	1	1	0	0	x/s ₁ = 600
0	s ₂	400	1	0	0	1	0	—
0	s ₃	300	0	1	0	0	1	300 →
	z _j	0	0	0	0	0	0	
	Δ _j	2	5	0	0	0	0	

C _B	Basic Variable	x _B	x	y	s ₁	s ₂	s ₃	Min Ratio
0	s ₁	300	1	0	1	0	0	x/s ₁ = 300 →
0	s ₂	400	1	0	0	1	0	= 400
5	y	300	0	1	0	0	1	—
	z _j	0	5	0	0	0	0	
	Δ _j	2	0	0	0	0	0	

C _B	Basic Variable	x _B	x	y	s ₁	s ₂	s ₃	Min Ratio
2	x	300	1	0	1	0	0	
0	s ₂	100	0	0	-1	1	0	
5	y	300	0	1	0	0	1	
	z _j	2	5	2	0	5		
	Δ _j	0	0	-2	0	-5		

$\rightarrow x=200, y=300$
 $\rightarrow \max Z=2100$

8/4/22

PROBLEM OF DEGENERACY

Sometimes the minimum ratio may not be unique. This causes the problem of degeneracy.

1. Tie in outgoing

For this type, take the unit matrix & divide by corresponding

Max $Z = 3x_1 + 9x_2$ s.t. $x_1 + 4x_2 \leq 8, x_1 + 2x_2 \leq 4, x_1, x_2 \geq 0$

std. form $\rightarrow \max Z = 3x_1 + 9x_2 + 0s_1 + 0s_2$

s.t. $x_1 + 4x_2 + s_1 = 8$ and $x_1 + 2x_2 + s_2 = 4$

Cj $\rightarrow 3 \quad 9 \quad 0 \quad 0$

C _B	Basic Variable	X _B	x ₁	x ₂	s ₁	s ₂	Min Ratio
0	s ₁	8	1	4	1	0	$x_8/x_2 = \frac{8}{4} = 2$
0	s ₂	4	1	2	0	1	2 $\rightarrow$
	Zj		0	0	0	0	
	Dj		3	9	0	0	

1st $\rightarrow$ $\begin{matrix} s_1 & s_2 \\ 1/4 & 0 \\ 0/2 & 1 \end{matrix}$
 min so row 2 is outgoing.

0	s ₁	0	-1	0	1	-2	
9	x ₂	2	1/2	1	0	1/2	
	Zj		9/2	9	0	9/2	
	Dj		-3/2	0	0	-9/2	

So, $x_1=0, x_2=2, \max Z=18$

2. Tie in Incoming Variable

It occurs when most two values of D_j are not unique.

Case 1 $\rightarrow$ when tie occurs betⁿ decision variables, then arbitrarily select any " ".

Case 2 $\rightarrow$ if there's a tie betⁿ slack or surplus variable, then, arbitrarily select any slack or surplus variable.

Case 3 $\rightarrow$ if there's a tie betⁿ a decision variable & slack/surplus variable, then always give priority to decision variable.

Unbounded solⁿ

Max $Z = 10x_1 + x_2 + 2x_3$ s.t. $14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + \frac{1}{2}x_2 - 6x_3 \leq 5, 3x_1 - x_2 - x_3 \leq 0, x_1, x_2, x_3 \geq 0$

std. form $\rightarrow \max Z = 10x_1 + x_2 + 2x_3 + 0x_4 + 0s_1 + 0s_2$

s.t. $14x_1 + x_2 - 6x_3 + 3x_4 = 7$

$16x_1 + \frac{1}{2}x_2 - 6x_3 + s_1 = 5$

$3x_1 - x_2 - x_3 + s_2 = 0$

Cj $\rightarrow 10 \quad 1 \quad 2 \quad 0 \quad 0 \quad 0$

C _B	Basic Variable	X _B	x ₁	x ₂	x ₃	x ₄	s ₁	s ₂	Min Ratio
0	x ₄	7	14	1	-6	3	0	0	$x_8/x_1 = \frac{7}{14} = \frac{1}{2}$
0	s ₁	5	16	1/2	-6	0	1	0	$\frac{5}{16} \rightarrow$
0	s ₂	0	3	-1	-1	0	0	1	
	Zj		0	0	0	0	0	0	
	Dj		10	1	2	0	0	0	
			$\uparrow$						
0	x ₄	2/8	0	1/16	-3/8	3	-7/8	0	
10	x ₁	5/16	1	1/32	-3/8	0	1/16	0	
0	s ₂	-15/16	0	-3/32	-1/8	0	-3/16	1	

$Z = 535$ 16	Z_j Δ_j	107 0	$\frac{107}{32}$ $-\frac{15}{32}$	$-\frac{32}{8}$ $33\frac{7}{8}$	0 0	$\frac{107}{16}$ $-\frac{107}{16}$	0 0
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So, solⁿ is unbounded

★ ALTERNATE OPTIMUM SOLⁿ

★ Big-M-Method / CHARNEC PENALTY METHOD

$\rightarrow 3x_1 + 2x_2 \geq 1$ artificial var.
 $3x_1 + 2x_2 + s_1 + a_1 = 1$ artificial var.
 $\rightarrow 2x_1 + 3x_2 \leq 1$
 $2x_1 + 3x_2 + s_2 = 1$ slack
 $x_1 + x_2 + a_2 = 1$ artificial var.

Note $\rightarrow$ Once artificial variable exist it can't enter again then stop calculating for that var.

① Solve the foll. LPP using Big-M Method

Max $Z = -2x_1 - x_2$

$s.t. \quad 3x_1 + x_2 = 3, \quad 4x_1 + 3x_2 \geq 6, \quad x_1 + 2x_2 \leq 4,$
 $x_1, x_2 \geq 0$

Std. form $\rightarrow$ Max $Z = -2x_1 - x_2 + 0s_1 + 0s_2 - Ma_1 - Ma_2$

$s.t. \quad 3x_1 + x_2 + a_1 = 3$
 $4x_1 + 3x_2 - s_1 + a_2 = 6$
 $x_1 + 2x_2 + s_2 = 4$

C_B	Basic var.	x_B	x_1	x_2	s_1	s_2	a_1	a_2	Min Ratio
-M	a_1	3	3	1	0	0	1	0	$\frac{3}{3} = 1 \rightarrow$
-M	a_2	6	4	3	-1	0	0	1	$\frac{6}{4}$
0	s_2	4	1	2	0	1	0	0	4
	Z_j		-1M	-4M	M	0	-M	-M	
	Δ_j		M-2	4M-1	-M	0	0	0	

5: Select the smallest quantity amongst the cells marked with

$\rightarrow$ std. form $Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - Ma_1$
 $s.t. \quad 2x_1 + x_2 + s_1 = 2$
 $3x_1 + 4x_2 - s_2 + a_1 = 12$

-2	x_1	1	1	$\frac{1}{3}$	0	0	0	0	3
-M	a_2	2	0	$\frac{1}{3}$	0	0	0	0	$\frac{3}{5} \rightarrow$
0	s_2	3	0	$\frac{5}{3}$	-1	0	0	0	$\frac{9}{5}$
	Z_j	-2	0	$-\frac{2}{3} - \frac{5M}{3}$	0	0	0	0	
	Δ_j	0	0	$\frac{1}{3} + \frac{5M}{3}$	-M	0	0	0	

-2	x_1	$\frac{3}{5}$	1	0	$\frac{1}{5}$	0	0	0	
-1	x_2	$\frac{6}{5}$	0	1	$-\frac{3}{5}$	0	0	0	
0	s_2	1	0	0	1	1	0	0	
	Z_j	-2	-1	0	0	0	0	0	
	Δ_j	0	0	$-\frac{1}{5}$	0	0	0	0	

$x_1 = \frac{3}{5}, x_2 = \frac{6}{5}, Z = -\frac{12}{5}$

Max $Z = 3x_1 + 2x_2$ s.t. $2x_1 + x_2 \leq 2, 3x_1 + 4x_2 \geq 12$
 $x_1, x_2 \geq 0$

C_B	Basic var.	x_B	x_1	x_2	s_1	s_2	a_1	Min Ratio
0	s_1	2	2	1	1	0	0	2 $\rightarrow$
-M	a_1	12	3	4	0	-1	1	3
	Z_j		-3M	-4M	0	M	-M	
	Δ_j		3+3M	4+4M	0	-M	0	
2	x_2	2	2	1	1	0	0	
-M	a_1	8	4	5	0	-4	-1	
	Z_j		4+5M	2	2+4M	M	-M	
	Δ_j		-1+5M	0	-2-4M	-M	0	

Here all Δ_j are ≤ 0 . So acc. to the optimality condⁿ this solⁿ is optimal. such solⁿ may be called pseudo-optimal solⁿ.

It means it doesn't satisfy all the constraints but it satisfies the optimality condⁿ of the Simplex method.

① Max $Z = x_1 + 2x_2 + 3x_3 - x_4$ s.t. $x_1 + 2x_2 + 3x_3 = 15$,
 $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$, $x_1, x_2, x_3, x_4 \geq 0$
 Using Big-M method.

Solⁿ std. form, Max $Z = x_1 + 2x_2 + 3x_3 - x_4 - M a_1 - M a_2$

s.t. $x_1 + 2x_2 + 3x_3 + a_1 = 15$

$2x_1 + x_2 + 5x_3 + a_2 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$\theta \rightarrow 1 \quad 2 \quad 3 \quad -1 \quad -M \quad -M$

C_B	Basic Var.	x_B	x_1	x_2	x_3	x_4	a_1	a_2	Min Ratio
-M	a_1	15	1	2	3	0	1	0	$\frac{15}{3} = 5$
-M	a_2	20	2	1	5	0	0	1	$\frac{20}{5} = 4 \rightarrow$
-1	x_4	10	1	2	1	1	0	0	10
$Z = -35M - 10$			$\theta_j \rightarrow -3M - 1$	$-3M - 2$	$-8M - 1$	-1	-M	-M	
			$\Delta_j \rightarrow 3M + 2$	$3M + 4$	$8M + 1$	0	0	0	

-M	a_1	3	$-\frac{1}{5}$	$\frac{7}{5}$	0	0	1	X	$\frac{15}{7} \rightarrow$
3	x_3	4	$\frac{2}{5}$	$\frac{1}{5}$	1	0	0	X	20
-1	x_4	6	$\frac{3}{5}$	$\frac{9}{5}$	0	1	0	X	$\frac{30}{9}$
$Z = -3M + 6$			$\theta_j \rightarrow \frac{M+3}{5}$	$\frac{-7M-6}{5}$	3	-1	-M	X	
			$\Delta_j \rightarrow \frac{2-M}{5}$	$\frac{1M+16}{5}$	0	0	0	X	

2	x_2	$\frac{15}{7}$	$-\frac{1}{7}$	1	0	0	X	X	$\frac{15}{7} \rightarrow$
3	x_3	$\frac{25}{7}$	$\frac{3}{7}$	0	1	0	X	X	$\frac{25}{7}$
-1	x_4	$\frac{15}{7}$	$\frac{6}{7}$	0	0	1	X	X	$\frac{15}{6} \rightarrow$
$Z = \frac{90}{7}$			$\theta_j \rightarrow \frac{1}{7}$	2	3	0	X	X	
			$\Delta_j \rightarrow \frac{15}{7}$	0	0	0	X	X	

2	x_2	$\frac{5}{2}$	0	1	0	$\frac{1}{6}$	X	X	
3	x_3	$\frac{5}{2}$	0	0	1	$\frac{1}{2}$	X	X	
1	x_1	$\frac{5}{2}$	1	0	0	$\frac{1}{6}$	X	X	
$Z = \frac{35}{2}$			$\theta_j \rightarrow 1$	2	3	$\frac{1}{6}$	X	X	
			$\Delta_j \rightarrow 0$	0	0	$-\frac{2}{3}$	X	X	

$\Rightarrow x_1 = x_2 = x_3 = \frac{5}{2}$, max $Z = 15$

② Use penalty method to max $Z = 3x_1 - x_2$ s.t.

$2x_1 + x_2 \geq 2$, $x_1 + 3x_2 \leq 3$, $x_2 \leq 4$, $x_1, x_2 \geq 0$

Solⁿ std. form $\rightarrow$ Max $Z = 3x_1 - x_2 + 0s_1 + 0s_2 - M a_1 + 0a_3 - M a_4$

s.t. $2x_1 + x_2 - s_1 + a_1 = 2$

$x_1 + 3x_2 + a_2 = 3$

$x_2 + s_2 = 4$

			$\theta \rightarrow 3$	-1	0	0	0	-M	
C_B	Basic Var.	x_B	x_1	x_2	s_1	s_2	a_1	a_2	Min Ratio
-M	a_1	2	2	1	-1	0	0	1	1 $\rightarrow$
0	s_2	3	1	3	0	1	0	0	3
0	s_3	4	0	1	0	0	1	0	
$Z = -2M$			$\theta_j \rightarrow -2M$	-M	M	0	0	-M	
			$\Delta_j \rightarrow 3+2M$	-1+M	-M	0	0	0	

3	x_1	1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	X	
0	s_2	2	0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	X	4 $\rightarrow$
0	s_3	4	0	1	0	0	1	X	

Z = 3	x_1	x_2	3	$\frac{3}{2}$	$-\frac{3}{2}$	0	0	X
	s_1	s_2	0	$-\frac{5}{2}$	$\frac{3}{2}$	0	0	X
					$\uparrow$			
Z = 9	x_1	x_2	3	1	3	0	1	0 X
	s_1	s_2	4	0	5	1	2	0 X
	s_3	s_4	4	0	1	0	0	1 X
Z = 9	x_1	x_2	3	9	0	3	0	X
	s_1	s_2	0	-10	0	-3	0	X

$s_0, x_1 = 3, x_2 = 0, \text{Max } Z = 9$

(3) Max. $Z = 4x_1 + 5x_2 - 3x_3 + 50$
 s.t. $x_1 + x_2 + x_3 = 10, x_1 - x_2 \geq 1, 2x_1 + 3x_2 + x_3 \leq 40$
 $x_1, x_2, x_3 \geq 0$

sol. std form $\rightarrow$ Max $Z = 4x_1 + 5x_2 - 3x_3 + 0s_1 + 0s_2 - M s_3 - M s_4$
 s.t. $x_1 + x_2 + x_3 + s_1 = 10$
 $x_1 - x_2 - s_2 = 1$
 $2x_1 + 3x_2 + x_3 + s_3 = 40$
 $x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0$

C_B	Basic Var.	x_B	x_1	x_2	x_3	s_1	s_2	s_3	s_4	Min Ratio
-M	s_1	10	1	1	1	0	0	1	0	10
-M	s_2	1	$\boxed{1}$	-1	0	-1	0	0	1	1 $\rightarrow$
0	s_3	40	2	3	1	0	1	0	0	20
Z = -4M										
	Z_j		-2M	0	-M	M	0	-M	-M	
	Δ_j		$4+2M$	5	$M+3$	-M	0	0	0	
			$\uparrow$							
-M	s_1	9	0	$\boxed{2}$	1	1	0	1	X	$\frac{9}{2} \rightarrow$
4	x_1	1	1	-1	0	-1	0	0	X	
0	s_2	38	0	5	1	2	1	0	X	$\frac{38}{5}$
Z = 9M + 4										
	Z_j		4	-2M+4	-M	-M+4	0	-M	X	
	Δ_j		0	$2M+9$	3M	M+4	0	0	X	

5	x_2	$\frac{9}{2}$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	X	X
4	x_1	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	X	X
0	s_3	$\frac{31}{2}$	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	X	X
Z = 89									
	Z_j		4	5	$\frac{1}{2}$	$\frac{1}{2}$	0	X	X
	Δ_j		0	0	$-\frac{15}{2}$	$-\frac{1}{2}$	0	X	X
$s_0, x_1 = \frac{11}{2}, x_2 = \frac{9}{2}, Z = \frac{89}{2} + 50$									
$= \frac{189}{2}$									

(4) Max $Z = 5x_1 - 2x_2 + 3x_3$ s.t. $2x_1 + 2x_2 - x_3 \geq 2, 3x_1 - 4x_2 \leq 3, x_2 + 3x_3 \leq 5$, and $x_1, x_2, x_3 \geq 0$
std form $\rightarrow$ Max $Z = 5x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 - M s_4 - M s_5$
 s.t. $2x_1 + 2x_2 - x_3 - s_1 + a_1 = 2, 3x_1 - 4x_2 + s_2 = 3, x_2 + 3x_3 + s_3 = 5$
 $x_1, x_2, x_3, s_1, s_2, s_3, s_4, s_5 \geq 0$

C_B	B.V.	x_B	x_1	x_2	x_3	s_1	s_2	s_3	s_4	s_5	Min Ratio
-M	a_1	2	$\boxed{2}$	2	-1	-1	0	0	1	0	1 $\rightarrow$
0	s_2	3	3	-4	0	0	1	0	0	0	
0	s_3	5	0	$\boxed{1}$	3	0	0	1	0	0	
Z = -2M											
	Z_j		-2M	-2M	M	M	0	0	-M	-M	
	Δ_j		$2M+5$	$2M-2$	$3-M$	-M	0	0	0	0	
			$\uparrow$								
5	x_1	1	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	X	X	
0	s_2	0	0	-7	$\boxed{\frac{3}{2}}$	$\frac{3}{2}$	1	0	X	X	0 $\rightarrow$
0	s_3	5	0	$\boxed{1}$	3	0	0	1	X	X	$\frac{5}{3}$
Z = 5											
	Z_j		5	5	$-\frac{5}{2}$	$-\frac{5}{2}$	0	0	X	X	
	Δ_j		0	-7	$\boxed{\frac{11}{2}}$	$\frac{5}{2}$	0	0	X	X	
					$\uparrow$						
5	x_1	1	1	$-\frac{4}{3}$	0	0	$\frac{1}{3}$	0	X	X	
3	x_3	0	0	$-\frac{11}{3}$	1	1	$\frac{2}{3}$	0	X	X	
0	s_3	5	0	$\boxed{15}$	0	3	0	1	X	X	$\frac{5}{15} \rightarrow$

$Z=5$		$Z_j \rightarrow$	5	$-\frac{6}{3}$	3	3	$\frac{1}{3}$	0	X
		$D_j \rightarrow$	0	$\frac{2}{3}$	0	-3	$-\frac{11}{3}$	0	X
5	x_1	$\frac{13}{9}$	1	0	0	$-\frac{4}{15}$	$\frac{7}{45}$	$\frac{4}{15}$	X
3	x_2	$\frac{14}{9}$	0	0	1	$\frac{1}{15}$	$\frac{2}{45}$	$\frac{14}{15}$	X
-2	x_2	$\frac{1}{3}$	0	1	0	$-\frac{1}{15}$	$-\frac{2}{45}$	$\frac{1}{15}$	X
$Z=10$		$Z_j \rightarrow$	5	-2	3	$-\frac{11}{15}$	$-\frac{53}{45}$	$-\frac{56}{45}$	X
9		$D_j \rightarrow$	0	0	0	$\frac{11}{15}$	$-\frac{53}{45}$	$-\frac{56}{45}$	X
5	x_1	$\frac{23}{3}$	1	0	4	0	$\frac{1}{3}$	$\frac{4}{3}$	X
0	δ_1	$\frac{70}{3}$	0	0	15	1	$\frac{2}{3}$	$\frac{14}{3}$	X
-2	x_2	5	0	1	3	0	0	1	X
$Z=85$		$Z_j \rightarrow$	5	-2	14	0	$\frac{5}{13}$	$\frac{14}{13}$	X
3		$D_j \rightarrow$	0	0	-11	0	$-\frac{5}{13}$	$-\frac{14}{13}$	X

$\therefore x_1 = \frac{23}{3}, x_2 = 5, x_3 = 0, \max Z = \frac{85}{3}$

5) Max $Z = 6x_1 + 4x_2$ s.t. $2x_1 + 3x_2 \leq 30, 3x_1 + 2x_2 \leq 24, x_1 + x_2 \geq 3$ and $x_1, x_2 \geq 0$.

solⁿ 1st form^s Max $Z = 6x_1 + 4x_2 + 0s_1 + 0s_2 + 0s_3 - M a_1$
s.t. $2x_1 + 3x_2 + s_1 = 30; 3x_1 + 2x_2 + s_2 = 24, x_1 + x_2 - s_3 + a_1 = 3$

		y_3	6	4	0	0	0	-M	
C_B	B.V.	x_B	x_1	x_2	s_1	s_2	s_3	a_1	Min Ratio
0	s_1	30	2	3	1	0	0	0	15
0	s_2	24	3	2	0	1	0	0	8
-M	a_1	3	1	1	0	0	-1	1	3 $\rightarrow$
$Z=3M$		Z_j	-M	-M	0	0	M	-M	
		D_j	6+M	4+M	0	0	-M	0	

0		δ_1	24	0	1	1	0	2	X	12
0		δ_2	15	0	-1	0	1	3	X	5 $\rightarrow$
6		x_1	3	1	1	0	0	-1	X	
$Z=18$		$Z_j \rightarrow$	6	6	0	0	0	-6	X	
		$D_j \rightarrow$	0	-2	0	0	0	6	X	
0		δ_1	14	0	$\frac{5}{3}$	1	$-\frac{2}{3}$	0	X	$\frac{14}{5} \rightarrow$
0		δ_2	5	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	X	
6		x_1	8	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	X	$\frac{8}{3} \rightarrow$
$Z=48$		$Z_j \rightarrow$	6	4	0	2	0	0	X	
		$D_j \rightarrow$	0	0	0	-2	0	0	X	

$x_1 = 8, x_2 = 0, Z = 48$

10/10/23

ALTERNATE OPTIMUM solⁿ

Max $Z = 6x_1 + 4x_2$ s.t. $2x_1 + 3x_2 \leq 30, 3x_1 + 2x_2 \leq 24, x_1 + x_2 \geq 3$ and $x_1, x_2 \geq 0$.
Is the solⁿ unique. If not give two diff. sol^s.

1st solⁿ

2nd solⁿ

in D_j
Here element corresponding to x_2 is 0
 $\Rightarrow$ Alternate solⁿ exists.

is me check kina h ki non-basic element ke D_j ke
corresponding j agr koi 0 h to alternate solⁿ exist

C_B	B.V.	x_B	x_1	x_2	s_1	s_2	s_3	a_1	Min Ratio
4	x_2	$\frac{42}{5}$	0	1	$\frac{3}{5}$	$-\frac{2}{5}$	0	X	
0	δ_2	$\frac{39}{5}$	0	0	$\frac{1}{5}$	$\frac{1}{5}$	1	X	
6	x_1	$\frac{12}{5}$	1	0	$-\frac{2}{5}$	$\frac{3}{5}$	0	X	
$Z=48$		$Z_j \rightarrow$	6	4	0	-2	0	X	
		$D_j \rightarrow$	0	0	0	2	0	X	

$\rightarrow \max Z = 48, x_1 = \frac{12}{5}, x_2 = \frac{42}{5}$

Q Min $Z = 2x_1 + 9x_2 + x_3$ s.t. $x_1 + 4x_2 + 2x_3 \geq 5$,
 $3x_1 + x_2 + 2x_3 \geq 4$, $x_1, x_2, x_3 \geq 0$.

sol. std form $\text{Max } Z' = -2x_1 - 9x_2 - x_3 + 0s_1 + 0s_2 - M a_1 - M a_2$

s.t. $x_1 + 4x_2 + 2x_3 - s_1 + a_1 = 5$

$3x_1 + x_2 + 2x_3 - s_2 + a_2 = 4$

$g_j \rightarrow -2 \quad -9 \quad -1 \quad 0 \quad 0 \quad -M \quad -M$

Q	B.V.	x_B	x_1	x_2	x_3	s_1	s_2	a_1	a_2	Min Ratio
-M	a_1	5	1	4	2	-1	0	1	0	$5/4 \rightarrow$
-M	a_2	4	3	1	2	0	-1	0	1	4
$Z = -9M$	Z_j	-4M	-5M	-4M	-M	M	M	-M	-M	
	Δ_j	$4M+2$	$5M+9$	$4M-3$	-M	-M	0	0	0	

-9	x_2	$5/4$	$1/4$	1	$1/2$	$-1/4$	0	X	0	5
-M	a_2	$1/4$	$1/4$	0	$3/2$	$1/4$	-1	X	1	$1 \rightarrow$

$Z = -45 - 11M$	Z_j	$-9 - 11M$	-9	$-9 - 3M$	$9 - 11M$	M	X	-M		
	Δ_j	$1 + 11M$	0	$3 + 3M$	$11M - 9$	-M	X	0		

-9	x_2	0	1	$1/11$	$-3/11$	$1/11$	X	X	$11/4$	
-2	x_1	1	1	0	$6/11$	$1/11$	-X	X	$11/6 \rightarrow$	

$Z = -11$	Z_j	-2	-9	$-24/11$	$-25/11$	$-1/11$	X	X		
	Δ_j	0	0	$31/11$	$-25/11$	$1/11$	X	X		

-9	x_2	$4/3$	$-2/3$	1	0	$-1/3$	$1/3$	X	X	$1 \rightarrow$
-1	x_3	$1/6$	$1/6$	0	1	$-2/3$	$-2/3$	X	X	-

$Z =$	Z_j	$25/6$	-9	-1	$17/6$	$-7/3$	X	X		
	Δ_j	$-37/6$	0	0	$-17/6$	$1/3$	X	X		

0	s_2	1	-2	3	0	-1	1	X	X	
-1	x_3	$5/2$	$1/2$	2	1	$-1/2$	0	X	X	

$Z = 5/2$	Z_j	$-1/2$	-2	-1	$7/2$	0	X	X		
	Δ_j	$-3/2$	-7	0	$-1/2$	0	X	X		

$x_1 = 0, x_2 = 0, x_3 = 5/2$

$\text{Max } Z = -5$

$\text{Min } Z = 5/2$

11/10/20

TWO PHASE SIMPLEX METHOD: convert into std form as Big-M method

Q Use 2-phase simplex method to solve the problem $\text{min } Z = x_1 - 2x_2 - 3x_3$

s.t. $-2x_1 + x_2 + 3x_3 = 2$, $2x_1 + 3x_2 + 4x_3 = 1$,
 $x_1, x_2, x_3 \geq 0$

sol. std form $\text{max } Z' = -x_1 + 2x_2 + 3x_3$

s.t. $-2x_1 + x_2 + 3x_3 - a_1 + a_2 = 2$

$2x_1 + 3x_2 + 4x_3 - s_2 + a_2 = 1$

PHASE-1 Auxiliary LPP is

$\text{Max } Z' = 0x_1 + 0x_2 + 0x_3 - a_1 - a_2$

Decision variable $\rightarrow x_1, x_2, x_3$
 Slack / surplus $\rightarrow s_1, s_2$
 artificial variable coeff $\rightarrow -1$
 Constraints same

$-2x_1 + x_2 + 3x_3 + a_1 = 2$

$2x_1 + 3x_2 + 4x_3 + a_2 = 1$

$x_1, x_2, x_3 \geq 0$

Phase I

Q	B.V.	x_B	x_1	x_2	x_3	a_1	a_2	Min Ratio
-1	a_1	2	-2	1	3	1	0	$2/3$
-1	a_2	1	2	3	4	0	1	$1/4 \rightarrow$
$Z = -3$	Z_j	0	-4	-7	-1	-1	-1	
	Δ_j	0	4	7	1	0	0	

$R_2 \rightarrow R_2 / 4$

$R_1 \rightarrow R_1 - 3R_2$

-1	a_1	$5/4$	$-7/2$	$-5/4$	0	1	$-3/4$
0	a_2	$1/4$	$1/2$	$3/4$	1	0	$1/4$
$Z^* = -5/4$	Z_j	$7/2$	$5/4$	0	-1	$3/4$	
	Δ_j	$-7/2$	$-5/4$	0	0	$-7/4$	

* Since all $\Delta_j \leq 0$ & optimum basic feasible solⁿ to the auxiliary LPP has been attained. But at the same time $\max Z^*$ is -ve & the artificial variable a_1 appears in the optimum basis at a +ve level. Hence, the ~~opt~~ original problem doesn't have any ^{feasible} physical solⁿ. here, there's no need to enter into phase 2.

→ Cond's observed after Phase 1-

① When $\max Z^* < 0$ & atleast 1 artificial variable appears in the optimum basis (B.V.) at a +ve level.

In this case, given problem doesn't have any solⁿ.

② When $\max Z^* = 0$ & atleast 1 artificial variable appears in the optimum basis (B.V.) at 0 level.

In this case proceed to phase 2.

③ $\max Z^* = 0$ & no artificial variable appears in the optimum basis.

In this case also proceed to phase 2.

$-1/2 + 1/2 = 0$
 $3/4 - 1/2 = 1/4$
 $-1/2 + 1/2 = 0$
 $3/4 - 1/2 = 1/4$

Q Min $Z = 15x_1 - 3x_2$ s.t. $3x_1 - x_2 - x_3 \geq 3$,
 $x_1 - x_2 + x_3 \geq 2$, $x_1, x_2, x_3 \geq 0$
Sol. Std form $\max Z' = -15x_1 + 3x_2$

s.t. $3x_1 - x_2 - x_3 - s_1 + a_1 = 3$,
 $x_1 - x_2 + x_3 - s_2 + a_2 = 2$.

Phase 1 Auxiliary LPP is

$\max Z^* = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - a_1 - a_2$

s.t. $3x_1 - x_2 - x_3 - s_1 + a_1 = 3$,
 $x_1 - x_2 + x_3 - s_2 + a_2 = 2$

$\text{RHS} \geq 0$ 0 0 0 0 0 -1 -1

C_B	B.V.	x_B	x_1	x_2	x_3	s_1	s_2	a_1	a_2	Min Ratio
-1	a_1	3	3	-1	-1	-1	0	1	0	1
-1	a_2	2	1	-1	1	0	-1	0	1	2
$Z^* = -5$	Z_j		-4	2	0	1	1	-1	-1	
	Δ_j		4	-2	0	-1	-1	0	0	
			$R_1 \rightarrow R_1 \div 3$	$\uparrow$	$R_2 \rightarrow R_2 - R_1$					
0	x_1	1	1	$-1/3$	$-1/3$	$-1/3$	0	$1/3$	0	-
-1	a_2	1	0	$-2/3$	$4/3$	$1/3$	-1	$-1/3$	1	$3/4 \rightarrow$
$Z^* = -1$	Z_j		0	$2/3$	$-4/3$	$-1/3$	1	$1/3$	-1	
	Δ_j		0	$-2/3$	$4/3$	$1/3$	-1	$-1/3$	0	
			$R_2 \rightarrow R_2 \times 3/4$	$\uparrow$	$R_1 \rightarrow R_1 + 3/4 R_2$					
0	x_1	$5/4$	1	$-1/2$	0	$-1/4$	$-1/4$	$1/4$	$1/4$	
0	x_2	$3/4$	0	$-1/2$	1	$1/4$	$-3/4$	$-1/4$	$3/4$	
$Z^* = 0$	Z_j		0	0	0	0	0	0	0	
	Δ_j		0	0	0	0	0	-1	-1	

all $\Delta_j \leq 0$ & no artificial variable in optimum basis.
So, we'll enter in phase 2.

Max Z's value →

Phase 2

CB	B.V.	x_B	x_1	x_2	x_3	s_1	s_2
0	x_1	5/4	1	-1/2	0	-1/4	-1/4
0	x_3	3/4	0	-1/2	1	1/4	-3/4
$Z' = -75$	Z_j	-15/2	15/4	0	15/8	15/8	
	Δ_j	0	-3/4	0	-15/8	-15/8	

Here all $\Delta_j \leq 0$. So, the optimum solⁿ has been attained.

$$\text{Max } Z' = -\frac{75}{8} \Rightarrow \text{Min } Z = \frac{75}{8}$$

$$x_1 = \frac{5}{4}, x_2 = 0, x_3 = \frac{3}{4}$$

Not for case 2

Phase 2 me jo A.V. bathega use phase 2 me bhi use krge or doosre ko hta denge.

Assignment

2-phase Method questions

① Max $Z = 3x_1 - x_2$ s.t. $2x_1 + x_2 \geq 2$, $x_1 + 3x_2 \leq 2$, $x_2 \geq 4$, and $x_1, x_2 \geq 0$.

Solⁿ Phase 1 → Auxiliary eqⁿ is

$$\text{Max } Z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - a_1 - a_2$$

$$\text{s.t. } 2x_1 + x_2 - s_1 + a_1 = 2$$

$$x_1 + 3x_2 + s_2 = 2$$

$$x_2 - s_3 + a_2 = 4$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Max Z's value →

Phase 2

CB	B.V.	x_B	x_1	x_2	s_1	s_2	s_3	a_1	a_2	Min Ratio
-1	a_1	2	2	1	-1	0	0	1	0	1 →
0	s_2	2	1	3	0	1	0	0	0	2
-1	a_2	4	0	1	0	0	-1	0	1	-
$Z' = -6$	Z_j	-2	-2	1	0	1	-1	-1	-1	
	Δ_j	2	2	-1	0	-1	0	0	0	
			↑							
0	x_1	1	1	1/2	-1/2	0	0	1/2	0	2
0	s_2	1	0	5/2	1/2	1	0	-1/2	0	2/5
-1	a_2	4	0	1	0	0	-1	0	1	4
$Z' = -4$	Z_j	0	-1	0	0	1	0	-1	-1	
	Δ_j	0	1	0	0	-1	-1	0	0	
			↑							
0	x_1	4/5	1	0	-3/5	-1/5	0	3/5	0	
0	x_2	2/5	0	1	1/5	2/5	0	-1/5	0	
-1	a_2	18/5	0	0	-1/5	-2/5	-1	1/5	1	
$Z' = -18/5$	Z_j	0	0	1/5	2/5	1	-1/5	-1	-1	
	Δ_j	0	0	-1/5	-2/5	-1	0	0	0	

Phase 2. No solⁿ exists.

② Max $Z = 5x_1 + 8x_2$ s.t. $3x_1 + 2x_2 \geq 3$, $x_1 + 4x_2 \geq 5$, $x_1 + x_2 \leq 5$, and $x_1, x_2 \geq 0$.

Solⁿ Phase 3 Auxiliary eqⁿ is

$$\text{Max } Z^* = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - a_1 - a_2$$

$$\text{s.t. } 3x_1 + 2x_2 - s_1 + a_1 = 3$$

$$x_1 + 4x_2 - s_2 + a_2 = 5$$

$$x_1 + x_2 + s_3 = 5$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

CB	B.V.	x_B	x_1	x_2	s_1	s_2	s_3	a_1	a_2	Min Ratio
-1	a_1	3	3	2	-1	0	0	-1	0	3/2
-1	a_2	4	1	4	0	-1	0	0	1	1 →
0	s_3	5	1	1	0	0	1	0	0	5
$Z^* = -1$	Z_j	-4	-6	1	1	0	-1	-1		
	Δ_j	4	6	-1	-1	0	0	0		

-1	a_1	1	5/6	0	-1	1/2	0	1	X	2/5 →
0	x_2	1	1/4	1	0	-1/4	0	0	1/4	4
0	s_3	4	3/4	0	0	1/4	1	0	-1/4	4 ÷ 3/4
$Z^* = -1$	Z_j	-5/2	0	1	-1/2	0	-1			
	Δ_j	5/2	0	-1	1/2	0	0			

0	x_1	2/5	1	0	-2/5	1/5	0	X	X	
0	x_2	9/10	0	1	1/10	-3/10	0	X	X	
0	s_3	8/10	0	0	3/10	1/10	1	X	X	
$Z^* = 0$	Z_j	0	0	0	0	0	0			
	Δ_j	0	0	0	0	0	0			

Max $Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$

Phase-2

CB	B.V.	x_B	x_1	x_2	s_1	s_2	s_3	Min Ratio
5	x_1	2/5	1	0	-2/5	1/5	0	2 →
8	x_2	9/10	0	1	1/10	-3/10	0	—
0	s_3	3/10	0	0	3/10	1/10	1	3/1
$Z = 4.6$	Z_j	5	8	-6/5	-7/5	0		
5	Δ_j	0	0	6/5	1/5	0		

0	s_2	2	5/6	0	-2	1	0	—		
8	x_2	15/10	15/20	1	-1/2	0	0	—		
0	s_2	25/10	25/2	0	1/2	0	1	7 →		
$Z = 10$	Z_j	12	8	-4	0	0				
	Δ_j	-7	0	4	0	0				

s_1	s_2	16	3	0	0	1	4			
s_2	x_2	5	1	1	0	0	1			
s_1	s_1	7	-1	0	1	0	2			
$Z = 40$	Z_j	8	8	0	0	0	8			
	Δ_j	-8	-8	0	0	0	-8			

so, Max $Z = 40$, $x_1 = 0$, $x_2 = 5$.

Q3 Max $Z = x_1 + 15x_2 + 2x_3 + 5x_4$

s.t. $3x_1 + 2x_2 + 4x_3 + x_4 \leq 6$,
 $2x_1 + x_2 + x_3 + 5x_4 \leq 4$,
 $2x_1 + 6x_2 - 8x_3 + 4x_4 = 0$,
 $x_1 + 3x_2 - 4x_3 + 3x_4 = 0$,
 $x_1, x_2, x_3, x_4 \geq 0$.

Step 1 Phase 1 Auxiliary eq's

Max $Z^* = 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0s_1 + 0s_2 - a_1 - a_2$
s.t. $3x_1 + 2x_2 + 4x_3 + x_4 + s_1 = 6$
 $2x_1 + x_2 + x_3 + 5x_4 + s_2 = 4$
 $2x_1 + 6x_2 - 8x_3 + 4x_4 + a_1 = 0$
 $x_1 + 3x_2 - 4x_3 + 3x_4 + a_2 = 0$

CB	B.V.	x_B	x_1	x_2	x_3	x_4	s_1	s_2	a_1	a_2	Min Ratio
0	s_1	6	3	2	4	1	1	0	0	0	3/3
0	s_2	4	2	1	1	5	0	1	0	0	4
-1	a_1	0	2	6	-8	4	0	0	1	0	0 →
-1	a_2	0	1	3	-4	3	0	0	0	1	0
$Z^* = 0$	Z_j	-3	-9	12	-7	0	0	-1	-1		
	Δ_j	3	9	-12	7	0	0	0	0		

0	x_2	3	3/2	1	2	1/2	1/2	0	0	0	6
0	s_2	1	1/2	0	-1	1/2	-1/2	1	0	0	2 →
-1	a_1	-18	-7	0	-20	1	3	0	1	0	—
-1	a_2	-9	-7/2	0	-10	3/2	-3/2	0	0	1	—

$\frac{15-2}{2} \rightarrow \frac{13}{2}$ $\frac{24}{-63} \rightarrow -12$ $3-2 \rightarrow 1$ $4-2 \rightarrow 2$ $-5+2 \rightarrow -3$ $-1+3 \rightarrow 2$
 $4+0 \rightarrow 4$ $12+0 \rightarrow 12$ $9-2 \rightarrow 7$ $2-2 \rightarrow 0$ $2-2 \rightarrow 0$ $1+1 \rightarrow 2$

5: Select the smallest quantity amongst the cells marked with

(NOTE) Agr phase 2 of the closed corresponding to the a.v. row. θ is always 0, for av. age to uske

$Z^0 = 27$

Z_j	$2/9$	0	30	-5	-3/2	0	-1	-1
Δ_j	-2/2	0	-30	5/2	3/2	0	0	0
X_2								
X_4	$2/9$	9	0	-2/9	1	-9	2/9	0
a_1	$-45/2$	-16	0					
a_2								

S_1	6	7/3	0	20/3	-1/3	1	0	0
S_2	4	5/3	0	7/3	13/3	0	1	0
X_2	0	1/3	1	-4/3	2/3	0	0	0
a_2	0	0	0	0	0	0	0	0
Z_j	0	0	0	0	-1	0	0	-1
Δ_j	0	0	0	0	1	0	0	0

S_1	6	7/3	0	20/3	0	1	0	2
S_2	4	5/3	0	7/3	0	0	1	2
X_2	0	1/3	1	-4/3	0	0	0	2
X_4	0	0	0	0	1	0	0	2
Z_j	0	0	0	0	0	0	0	2
Δ_j	0	0	0	0	0	0	0	2

Phase 2 →

Max $Z' = x_1 + 15x_2 + 2x_3 + 5x_4 + 0x_5 + 0x_6$

G_j	G_j	1	15	2	5	0	0
B.V.	X_B	x_1	x_2	x_3	x_4	s_1	s_2
0	S_1	6	7/3	0	20/3	0	0
0	S_2	4	5/3	0	7/3	0	0
15	x_2	0	1/3	1	-4/3	0	0
5	x_4	0	0	0	0	1	0
Z_j		5	15	-20	5	0	0
Δ_j		4	0	22	0	0	0

$Z^1 = 19.8$

Z_j	2.89	12.7	15	0	1	0	0
Δ_j	-2.89	12.7	15	0	2	0	0
Max Z'	19.8						

$x_1 = 0, x_2 = 1.2, x_3 = 0.9, x_4 = 0$

Q4 Minimize $Z = x_1 - 2x_2 - 3x_3$ s.t. $-2x_1 + x_2 + 3x_3 = 2$
 $2x_1 + 3x_2 + 4x_3 = 1, x_1, x_2, x_3 \geq 0$

Std form $\rightarrow$ Max $Z = -x_1 + 2x_2 + 3x_3 - a_1 - a_2$
 Phase 1 Auxiliary eq $\rightarrow$ Max $Z' = 0x_1 + 0x_2 + 0x_3 - a_1 - a_2$
 s.t. $-2x_1 + x_2 + 3x_3 + a_1 = 2$
 $2x_1 + 3x_2 + 4x_3 + a_2 = 1$

C_B	B.V.	x_B	x_1	x_2	x_3	a_1	a_2	Min Ratio
-1	a_1	2	-2	1	1	1	0	
-1	a_2	1	2	3	4	0	1	
$Z = -3$								

Done Before in C/O

Q5 Max $Z = 3x_1 + 2x_2 + x_3 + 4x_4$ s.t.
 $4x_1 + 5x_2 + x_3 - 3x_4 = 5, 2x_1 - 3x_2 - 4x_3 + 5x_4 = 7$
 $x_1 + 4x_2 + 2.5x_3 - 4x_4 = 6, x_1, x_2, x_3 \geq 0$

Phase 1
 Auxiliary eq $\rightarrow$
 Max $Z' = 0x_1 + 0x_2 + 0x_3 + 0x_4 - a_1 - a_2 - a_3$
 s.t. $4x_1 + 5x_2 + x_3 - 3x_4 + a_1 = 5$
 $2x_1 - 3x_2 - 4x_3 + 5x_4 + a_2 = 7$
 $x_1 + 4x_2 + 2.5x_3 - 4x_4 + a_3 = 6$

CB	B.V.	x_0	x_1	x_2	x_3	x_4	a_1	a_2	a_3	Min Ratio
-1	a_1	5	1	2	1	-3	1	0	0	1
-1	a_2	7	2	-3	-4	5	0	1	0	1
-1	a_3	6	1	4	2.5	-4	0	0	1	1
$Z^* = -18$	Z_j	-7	-12	-5	2	-1	-1	-1		
	D_j	7	12	5	-2	0	0	0		

0	x_1	1	4/5	1	1/5	-3/5	X	0	0	-
-1	a_1	4	2/5	0	-2/5	34/9	X	1	0	→
-1	a_3	2	-1/5	0	1/5	-8/5	X	0	1	-
$Z^* = -6$	Z_j	13/5	0	2.5	-48/45	X	-1	-1		
	D_j	-13/5	0	-2.5	48/45	X	0	0		

0	x_2	139/85	.86	1	-.87	0	X	X	0	2
0	x_4	18/17	9/85	0	-189/120	1	X	X	0	
-1	a_3	3.69				0	X	X	1	
$Z^* = -3.69$	Z_j					0	X	X	-1	
	D_j					0	X	X	0	

0	x_1	1.25	1	1.25	0.25	-0.75	0	0	-	
-1	a_2	4.5	0	-5.5	-4.5	6.5	1	0	→	
-1	a_3	4.75	0	2.75	2.25	-3.25	0	1	-	
$Z^* = -9.5$	Z_j	0	2.75	2.25	-3.25	-1	-1			
	D_j	0	-2.75	-2.25	3.25	0	0			

0	x_1	1.76	1	.62	-.27	-0.26	0	2.0		
0	x_4	.69	0	-.85	-.70	-0.69	1	2.0		
-1	a_3	7	0	0	0	0	0	2.1		
$Z^* = -7$	Z_j	0	0	0	0	0	0	2.1		
	D_j	0	0	0	0	0	0	2.0		

NO Soln

⑥ Max $Z = 5x_1 - 2x_2 + 3x_3$ s.t. $2x_1 + x_2 - x_3 \geq 2$
 $3x_1 - 4x_2 \leq 3 \rightarrow x_2 + 3x_3 \leq 5, x_1, x_2, x_3 \geq 0$
 Phase 1 → A.E. Max $Z' = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 + 0s_3 - a_1$
 s.t. $2x_1 + x_2 - x_3 - s_1 + a_1 = 2, 3x_1 - 4x_2 + s_2 = 3$
 $x_2 + 3x_3 + s_3 = 5$

CB	B.V.	x_0	x_1	x_2	x_3	s_1	s_2	s_3	a_1	Min
-1	a_1	2	2	2	-1	-1	0	0	1	1 →
0	s_2	3	3	4	0	0	1	0	0	1
0	s_3	5	0	2	3	0	0	1	0	-
$Z^* = -2$	Z_j	-2	-2	1	1	0	0	0	-1	
	D_j	2	2	-1	-1	0	0	0	1	

0	x_1	1	1	1	-1/2	-1/2	0	0	1	→
0	s_2	0	0	-7	3/2	3/2	1	0	1	
0	s_3	5	0	12	3	0	0	12	1	
$Z^* = 0$	Z_j	0	0	0	0	0	0	0	0	
	D_j	0	0	0	0	0	0	0	0	

Phase 2 → A.E. Max $Z' = 5x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3$

CB	B.V.	x_0	x_1	x_2	x_3	s_1	s_2	s_3	Min
5	x_1	1	1	1	$-1/2$	$-1/2$	0	0	→
0	s_2	0	0	-7	$3/2$	$3/2$	1	0	0
0	s_3	5	0	12	3	0	0	12	$5/2$
$Z' = 5$	Z_j	5	5	$-9/2$	$-9/2$	0	0	0	
	D_j	0	0	7	$11/2$	$9/2$	0	0	
					↑				

5	x_1	1	1	-4/3	0	0	1/3	0	-	
3	x_3	0	0	14/3	1	1	2/3	0	-	
0	s_3	5	0	0	0	-3	-2	1	5	→

$Z=5$	Z_j	5	10 3	3	3	$1/2$	0
	D_j	0	$1/2$	0	-3	$-1/3$	0
5	x_1	13/9	1	0	0	$1/45$	$4/45$
3	x_2	14/9	0	0	1	$1/45$	$14/45$
2	x_3	$1/3$	0	1	0	$-1/5$	$1/5$
$Z=$	Z_j	5	-2	3	$-12/5$	$63/45$	$58/45$
	D_j	0	0	0	$12/5$	$-63/45$	$-58/45$
5	x_1	69	1	0	4	0	$1/3$
10	x_2	210	0	0	0	1	$2/3$
-2	x_3	5	0	1	$1/5$	0	0
$Z=$	Z_j	5	-2	14	0	$5/3$	$14/3$
	D_j	0	0	-11	0	$-5/3$	$-14/3$

So, $Z' = \frac{85}{3}$, $x_1 = \frac{69}{9}$, $x_2 = 5$, $x_3 = 0$

Q7 Max $Z = 2x_1 + 3x_2 + 5x_3$ s.t. $3x_1 + 10x_2 + 5x_3 \leq 15$,
 $x_1 + 2x_2 + x_3 \geq 4$, $33x_1 - 10x_2 + 9x_3 \leq 33$, $x_1, x_2, x_3 \geq 0$

Phase I $\rightarrow$ A.E. $\rightarrow$ Max $Z' = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 + 0s_3 - a_1$

s.t. $3x_1 + 10x_2 + 5x_3 + s_1 = 15$

$x_1 + 2x_2 + x_3 - s_2 + a_1 = 4$

$33x_1 - 10x_2 + 9x_3 + s_3 = 33$

	C_B	B.V.	x_1	x_2	x_3	s_1	s_2	s_3	a_1	Min
0	s_1	15	3	10	5	1	0	0	0	$15/10 \rightarrow$
-1	a_1	4	1	2	1	0	-1	0	1	$4/2$
0	s_3	33	33	-10	9	0	0	1	0	-
$Z=4$	Z_j	-1	-2	-1	0	1	0	-1	-1	
	D_j	1	2	1	0	-1	0	0	0	

0	x_2	$3/2$	$3/10$	1	$1/2$	$1/10$	0	0	0
-1	a_1	1	$2/5$	0	0	$-1/5$	-1	0	1
0	s_3	18	36	0	4	1	0	1	0
$Z'=-1$	Z_j	$-2/5$	0	0	$1/5$	1	0	-1	
	D_j	$2/5$	0	0	$-1/5$	-1	0	0	
0	x_2	$4/6$	0	1	.38	-.09	0	-.08	0
-1	a_1	$7/15$	0	0	$-1/45$	-.21	-1	$1/36$	1
0	x_1	$4/3$	1	0	$1/8$	.02	0	$1/36$	0
$Z=-2$	Z_j	0	0	$1/45$	-.21	1	$1/36$	-1	
	D_j	0	0	$-1/45$	-.21	-1	$1/36$	0	

NO solⁿ exist.

Q8 Max $500x_1 + 1400x_2 + 900x_3$ s.t.

$x_1 + x_2 + x_3 = 100$, $12x_1 + 35x_2 + 15x_3 \geq 25$,

$8x_1 + 3x_2 + 4x_3 \geq 6$, $x_1, x_2, x_3 \geq 0$

Phase I $\rightarrow$ A.E. $\rightarrow$ Max $Z' = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - a_1 - a_2$
 s.t. $x_1 + x_2 + x_3 + a_1 = 100$; $12x_1 + 35x_2 + 15x_3 - s_1 + a_2 = 25$

$8x_1 + 3x_2 + 4x_3 - s_2 + a_3 = 6$

CB	B.V.	x_1	x_2	x_3	s_1	s_2	a_1	a_2	a_3	Min
0	a_1	100	1	1	1	0	0	1	0	100
0	a_2	25	12	35	15	-1	0	0	1	$25/15 \rightarrow$
0	a_3	6	8	3	4	0	-1	0	1	3
$Z=131$	Z_j	-21	-39	-20	1	1	-1	-1	-1	
	D_j	21	39	20	-1	-1	0	0	0	
-1	a_1	$6/5$	$23/35$	0	$1/4$	$1/35$	0	1	0	1.51.08
0	x_2	$5/7$	$12/35$	1	$3/7$	$-1/35$	0	0	1	2.08
-1	a_3	$3/85$	$6/97$	0	$3/114$	.085	-1	0	1	0.55
$Z'=-103.14$	Z_j	-7.63	0	-3.31	-.114	1	-1	0	-1	
	D_j	7.63	0	3.31	.114	-1	0	0	0	

-1	a_1	98.92	0	0	-315	-0.0205	0.094	1	✓
0	a_2	0.52	0	1	-295	-0.0328	0.05	0	✓
0	x_1	0.5	1	0	-389	-0.0122	-0.14	0	✓
$Z = -98.92$		Z_j	0	0	-315	-0.0205	-0.094	-1	
		Δ_j	0	0	-315	0.0205	0.094	0	

-1	a_1								
0	x_2								
0	x_3	1.413	2.57	0	1	-0.0316	-359	0	

Max. $Z = 5x_1 + 2x_2$ st. $6x_1 + x_2 \geq 6$, $4x_1 + 3x_2 \geq 12$, $x_1 + 2x_2 \geq 4$, $x_1, x_2 \geq 0$.

Phase 1: Max $Z' = 0x_1 + 0x_2 + 0s_1 + 0s_2 + 0s_3 - a_1 - a_2 - a_3$
 st. $6x_1 + x_2 - s_1 + a_1 = 6$
 $4x_1 + 3x_2 - s_2 + a_2 = 12$
 $x_1 + 2x_2 - s_3 + a_3 = 4$

a_1	B.V.	x_2	x_1	x_2	s_1	s_2	s_3	a_1	a_2	a_3	Min Ratio
-1	a_1	6	6	1	-1	0	0	1	0	0	1 →
-1	a_2	12	4	3	0	-1	0	0	1	0	3
-1	a_3	4	1	2	0	0	-1	0	0	1	4
$Z' = -22$		Z_j	-11	-6	1	1	1	-1	-1	-1	
		Δ_j	11	6	-1	-1	-1	0	0	0	

0	x_1	1	1	1/6	-1/6	0	0	1	0	0	6
-1	a_2	8	0	7/3	2/3	-1	0	1	1	0	24/7
-1	a_3	3	0	1/6	1/6	0	-1	1	0	1	18/11 →
$Z' = -11$		Z_j	0	-25/6	1/2	1	1	-1	-1	-1	
		Δ_j	0	25/6	-1/2	-1	-1	1	0	0	

0	x_1	5/11	1	0	-2/11	0	0	1	0	0	
-1	a_2	48/11	0	0	-1/11	-1	1	1	1	1	
0	x_2	18/11	0	1	1/11	0	-6/11	1	0	0	
$Z' = -138/11$		Z_j	0	0	19/18	1	-14/11	-1	-1	-1	
		Δ_j	0	0	-19/18	-1	14/11	0	0	0	

Phase 2: Max $Z' = 5x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3$

a_1	B.V.	x_2	x_1	x_2	s_1	s_2	s_3	Min Ratio
5	x_1	0.428	18	0	-214	0.071	0	—
0	s_1	3.28	-10	0	-13.357	-7.85	1	9.2 →
2	x_2	3.286	-60	1	13.205	-4.86	0	12
$Z' = 52.9$		Z_j	43.5	2	13.6-0.5	-0.5	0	
		Δ_j	-38.0	0	-13.6	0.5	0	

Max $Z = 52$, $x_1 = 8$, $x_2 = 6$

5	x_1	24	1	0	0	-4	0.6	—
0	s_1	9.2	0	0	1	-2.2	2.8	4 →
2	x_2	0.8	0	1	0	0.2	-0.8	
$Z' = 13.6$		Z_j	5	2	0	-1.6	1.4	
		Δ_j	0	0	0	1.6	-1.4	

CB	B.V.	x_B	x_1	x_2	s_1	s_2	s_3	Min
0	s_1	-3	-3	-1	1	0	0	
0	s_2	-6	-4	-3	0	1	0	
0	s_3	-3	-1	-2	0	0	1	
0	$Z=0$	Z_j	0	0	0	0	0	
	D_j		-2	-1	0	0	0	

here all D_j 's & x_B 's are -ve. $\Rightarrow$ s_3 is non-feasible
so we'll make x_3 +ve or 0.

- Take most -ve element of x_B as outgoing.
- * D_j 's by corresponding outgoing Row (take only -ve's)
- Choose min of all D_j 's.
- Min Element corresponding to min D_j & outgoing one is key element. (intersection)

$-2/4$ $(-1/2) \rightarrow$ min.

0	s_1	-1	$(-5/3)$	0	1	$-1/3$	0	
-1	x_2	2	$4/3$	1	0	$-1/3$	0	
0	s_3	1	$5/3$	0	0	$-2/3$	1	
$Z=-2$	Z_j	$-4/3$	$-10/3$	-1	0	$1/3$	0	
	D_j	$-2/3$	$-2/3$	0	0	$-1/2$	0	
	D_j/s_3	$(2/3)/(-5/3)$				$-1/3/-1/3$	0	

-2	x_1	$3/5$	1	0	$-3/5$	$1/5$	0	
-1	x_2	$6/5$	0	1	$4/5$	$-3/5$	0	
0	s_3	0	0	0	1	-1	1	
$Z=$								

Q2 Min $Z = x_1 + 2x_2$ s.t. $2x_1 + x_2 \geq 4$, $x_1 + 2x_2 = 7$,
 $x_1, x_2 \geq 0$

So $\Rightarrow$ Max $Z' = -x_1 - 2x_2 + 0s_1 + 0s_2$
s.t. $-2x_1 - x_2 + s_1 = -4$
 $-x_1 - 2x_2 + s_2 = -7$

CB	B.V.	x_B	x_1	x_2	s_1	s_2	Min
0	s_1	-4	-2	-1	1	0	
0	s_2	-7	-1	-2	0	1	
$Z'=0$	Z_j	0	0	0	0	0	
	D_j		-1	-2	0	0	
	D_j/s_2		1	1	-	-	

0	s_1	-10	0	3	1	-2	
-1	x_2	7	1	2	0	-1	
$Z'=-7$	Z_j	-1	-2	0	0	1	
	D_j	0	0	0	0	-1	
	D_j/s_2						

$\Rightarrow x_1=0, x_2=7$ Max $Z'=-7$
 $\Rightarrow$ Min $Z=7$

Q3 Max $Z = -4x_1 - 6x_2 - 18x_3$ s.t. $x_1 + 3x_3 \geq 3$,
 $x_2 + 2x_3 \geq 5$ and $x_1, x_2, x_3 \geq 0$

So $\Rightarrow$ Max $Z = -4x_1 - 6x_2 - 18x_3 + 0s_1 + 0s_2$
s.t. $-x_1 - 3x_3 + s_1 = -3$
 $-x_2 - 2x_3 + s_2 = -5$

CB	B.V.	x_B	x_1	x_2	x_3	s_1	s_2	Min
0	s_1	-3	-1	0	-3	1	0	
0	s_2	-5	0	-1	-2	0	1	
$Z=0$	Z_j	0	0	0	0	0	0	
	D_j		-4	-6	-18	0	0	
	D_j/s_2			6	9	-	-	

C_B	B.V.	x_B	x_1	x_2	x_3	s_1	s_2	Min
0	s_1	-3	-1	0	-3	1	0	→
-6	x_2	5	0	1	2	0	-1	
$Z = -30$	Z_j	0	-6	-12	0	0	6	
	D_j	-4	0	-6	0	-6		
	A_j/s_1	4	-	2	-	-		
								min{2,4}
-18	x_3	1	1/3	0	1	-1/3	0	
-6	x_2	3	-2/3	1	0	2/3	-1	
$Z = -36$	Z_j	-2	-6	-18	2	6		
	D_j	-2	0	0	-2	-6		

Max $Z = -36$, $x_1 = 0$, $x_2 = 3$, $x_3 = 1$

Q4 Min $Z = 6x_1 + x_2$ s.t. $2x_1 + x_2 \geq 3$, $x_1 - x_2 \geq 0$
 Max $Z' = -6x_1 - x_2 + 0s_1 + 0s_2$
 s.t. $-2x_1 - x_2 + s_1 = -3$
 $-x_1 + x_2 + s_2 = 0$

C_B	B.V.	x_B	x_1	x_2	s_1	s_2	Min
0	s_1	-3	-2	-1	1	0	→
0	s_2	0	-1	1	0	1	
$Z' = 0$	Z_j	0	0	0	0	0	
	D_j	-6	-1	0	0		
	A_j/s_1	3	1	-	-		
							min{2,1}
-1	x_2	3	2	1	-1	0	
0	s_2	-3	-1	0	1	1	→
$Z' = -3$	Z_j	-2	-1	0	0		
	D_j	-4	0	-1	0		
	A_j/s_2	4/3	-	-	-		

-1	x_2	1	0	1	-1/3	2/3
-6	x_1	1	1	0	-1/3	-1/3
$Z' = -7$	Z_j	-6	-1	2/3	8/3	
	D_j	0	0	-7/3	-8/3	

Max $Z' = -7 \rightarrow$ Max $Z = 07$, $x_1 = x_2 = 1$

18/8/23

UNIT-II

ASSIGNMENT PROBLEM

Suppose there are n jobs to be performed & n persons are available for doing these jobs. Assume that each person can do each job at a time though with varying degree of efficiency. Let c_{ij} be the cost (payment) if the i th person is assigned the j th job. The problem is to find the assignment so that total cost for performing all jobs is minimum.

	Jobs.							
	1	2	3	...	j	...	n	
persons	1	c_{11}	c_{12}	c_{13}	...	c_{1j}	...	c_{1n}
	2	c_{21}	c_{22}	c_{23}	...	c_{2j}	...	c_{2n}
	3	c_{31}						
	c_{i1}	c_{i2}	c_{i3}	...				
	c_{n1}	c_{n2}	c_{n3}	...				

Ex 18 Assigning classes to rooms,
" drivers " tractors,
" men to jobs "

Mathematical Formulation of Assignment Problem
Mathematically the assignment problem can be stated as

$$\text{Minimize the total cost } Z = \sum_{i=1}^n \sum_{j=1}^n C_{ij} x_{ij}$$

subject to the restrictions of the form
 $x_{ij} = \begin{cases} 1, & \text{if } i\text{th person is assigned to } j\text{th job.} \\ 0, & \text{otherwise.} \end{cases}$

$$\sum_{j=1}^n x_{ij} = 1 \quad (1 \text{ job is done by the } i\text{th person } i=1, \dots, n)$$

$$\sum_{i=1}^n x_{ij} = 1 \quad (\text{only 1 person should be assigned the } j\text{th job} \\ \text{+ job is done by the } i\text{th person})$$

HUNGARIAN METHOD (to solve assignment problem)

Step 1) a) If the no of sources = no of destinations.

Then, go to step 3.

b) If no of sources $\neq$ no of destinations. Then, go to step 2.

Step 2) Add a dummy source or dummy destination so that the cost table becomes a square matrix. The cost entries of dummy sources/destinations are always zero.

Step 3) Find out the smallest element in each row of the given matrix & then subtract the same from each column of that row.

Step 4) Find out the smallest element in each column & then subtract the same from each element of

that row.

Step 5) In the modified matrix of step 4 do the optimal assignment.

i) Examine the rows until a row with single zero is obtained. Mark this zero as (0) and cross out (X) all other zeros in that column.

Continue in this manner until all the rows have been considered.

ii) Repeat this procedure for each column of the reduced matrix.

iii) If a row or column has two or more zeros & 1 can't be assigned then assign arbitrary any one of these zeros and cross out all other zeros of that row/column.

iv) Repeat 1st point to 3rd point successively until the chain of assigning complete.

Step 6) If the no of assignments = n, then an optimal sol is obtained.

Q. Solve the minimization assignment problem

	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Sol Step 1) It is a balanced assignment ($\because$ no of tasks = no of persons)

Step 2) Subtract the smallest element of each row from each element of that row.

First reduced matrix

Persons

Tasks	I	II	III	IV
A	0	18	9	3
B	9	24	0	22
C	23	4	3	0
D	9	16	14	0

Step 4: Subtract smallest from each column
2nd reduced matrix

Persons

Tasks	I	II	III	IV
A	0	14	9	3
B	9	20	0	22
C	23	0	3	0
D	9	12	14	0

Step 6: here no. of marked zeroes (0) = no. of tasks =
no. of persons = 4.

So, optimal assignment has been attained.

Now, for minimum cost:

A - I	8
B - III	4
C - II	19
D - IV	10
	<u>41</u>

Step 7 [Optimality Criteria]

a) If all zero elements in the ^{cost} matrix either marked 0 or are crossed (x) and there is exactly one assignment in each row and column, then it is optimal sol.

If total cost associated with this sol is obtained by adding the original cost elements in the occupied cells.

- b) If a zero element in a row or column were chosen arbitrarily for assignment in step 5, $\exists$ an alternate optimal sol.
- c) If there is no assignment in a row (or column) then $\Rightarrow$ total no. of assignments are less than the no. of rows/columns in a square matrix.

Step 8 [Revise the opportunity cost matrix]

Draw a set of horizontal & vertical lines to cover all the zeroes in the revised cost matrix obtained from the step 5 by using the following procedure:-

- a) For each row in which no assignment are made mark a tick (✓).
- b) Examine the marked row, if any zero element is present in these rows mark a tick (✓) to the respective columns containing zeroes.
- c) Examine marked columns, if any assigned zero element is present in these columns tick (✓) the respective rows containing assigned zeroes.
- d) Repeat this process until no more rows or columns can be marked.
- e) Draw a straight line through each marked column & unmarked row.

If the no. of lines drawn = no. of rows/columns then, the current sol is optimal sol. Otherwise go to step 1.

Step 9: Develop a new revised opportunity cost matrix.

- Among the elements in the matrix not covered by any line, choose the smallest element & call this value 'K'.
- Subtract K from every element in the matrix that is not covered by a line.
- Add K to every element in the matrix covered by 2 lines i.e. intersection of 2 lines.
- Element in the matrix covered by 1 line remains unchanged.

Step 10: Repeat step 5-9 until an optimal solⁿ is obtained.

Q Program →

Programmer	A	B	C
1	120	100	80
2	80	90	110
3	110	140	120

solⁿ

	A	B	C
1	40	20	0
2	0	10	30
3	0	30	10

2nd solⁿ

	A	B	C
1	40	10	0
2	X	0	30
3	0	20	10

optimal solⁿ obtained

1 - A	80
2 - B	90
3 - C	110
	<u>280</u>

Q

	I	II	III	IV	V
A	10	5	13	15	16
B	3	9	13	13	6
C	10	7	2	2	2
D	7	11	9	7	12
E	7	9	10	4	12

1st solⁿ

	I	II	III	IV	V
A	5	0	8	10	11
B	0	6	5	10	3
C	8	5	0	X	X
D	7	4	2	X	5
E	7	5	6	0	8

unticked rows & ticked column

2nd solⁿ

	I	II	III	IV	V
A	3	0	4	11	10
B	0	2	9	9	0
C	3	0	0	0	0
D	0	4	0	3	6
E	0	2	1	0	6

	I	II	III	IV	V
A	7	0	8	12	11
B	0	4	13	10	1
C	10	5	X	2	0
D	X	2	0	X	3
E	3	3	2	0	6

optimal.

A → II	5
B → I	3
C → V	2
D → III	9
E → IV	4
	<u>+ 23</u>

	P	Q	R	S	
A	85	75	65	125	75
B	90	78	66	132	78
C	75	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

	P	Q	R	S	T
A	20	10	0	60	10
B	24	12	0	66	12
C	18	9	0	57	12
D	20	12	0	60	12
E	20	8	0	56	12

	P	Q	R	S	T
A	2	2	X	4	10
B	6	4	10	10	2
C	10	1	X	1	2
D	2	4	X	4	2
E	2	10	X	X	2

	P	Q	R	S	T
A	2	2	2	24	X
B	4	2	10	8	X
C	10	1	2	1	2
D	X	2	X	2	10
E	2	10	2	X	2

	P	Q	R	S	T
A	2	2	2	4	10
B	4	2	10	8	X
C	10	1	2	1	2
D	X	2	X	2	X
E	2	10	2	X	X

	P	Q	R	S	T
A	2	1	2	3	10
B	4	1	10	7	X
C	10	X	2	2	2
D	X	1	X	1	10
E	3	10	3	X	3

	P	Q	R	S	T
A	2	1	2	2	X
B	4	X	10	6	X
C	0	-1	2	-1	2
D	0	0	0	0	0
E	4	0	4	0	4

	P	Q	R	S	T
A	2	1	2	3	10
B	4	1	10	7	X
C	X	10	2	X	2
D	10	1	X	1	X
E	3	X	3	10	3

A → T → 75
 B → R → 66
 C → Q → 66
 D → P → 80
 E → S → 112

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24/08/23

VARIATIONS OF THE ASSIGNMENT PROBLEM

* Multiple Optimal Solⁿ

While making an assignment in the reduced assignment matrix it is possible to have

2 or more ways to strike off a certain no. zeroes.

Such a situation indicates that there are multiple optimal solⁿ with the same optimal value of the objective funⁿ.

* Maximization Case in Assignment Problem

If instead of cost matrix, a profit matrix is given then assignments are made in such a way that total profit is maximized.

The profit maximization problems are solved by converting them into a cost minimization problem in either of the following 2 ways:-

- ① Put a -ve sign before each of the element in the profit matrix, in order to convert profit value into a cost value.
- ② Locate the largest element in the profit matrix & subtract all the elements of the matrix from the largest element including itself.

Q Maximize the sales

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35 38	39

Sales man

	A	B	C	D	E
1 st 1	9	14	9	21	1
2	1	25	21	28	13
3	0	22	16	19	12
4	27	11	8	13	13
5	20	16	9	14	10

	A	B	C	D	E
1 st 1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	8	1	6	2

	A	B	C	D	E
2 nd 1	8	2	0	12	0
2	0	16	12	19	4
3	0	14	8	11	4
4	19	3	0	5	5
5	11	7	0	5	1

	A	B	C	D	E
3 rd 1	8	16	8	7	8
2	0	14	12	14	4
3	8	12	8	6	4
4	19	1	0	8	5
5	11	5	8	0	1

	A	B	C	D	E
4 th 1	12	0	8	7	8
2	0	10	8	10	4
3	8	8	4	2	0
4	23	1	0	8	5
5	15	5	8	0	1

1 → B → 28	1	A	B	C	D	E
2 → A → 40	2	12	10	8	10	7
3 → E → 37	3	8	8	4	2	10
4 → C → 41	4	23	1	8	10	7
5 → D → 35	5	15	5	10	7	1
<u>191</u>						

1 → B → 38
2 → A → 40
3 → E → 27
4 → D → 36
5 → C → 40

191

→ Multiple optimal solⁿ.

Unbalanced Assignment Problem

The Hungarian method for solving assignment problem requires that no. of rows and columns in assignment matrix should be equal.

However, when the cost matrix is not a square matrix. The assignment problem is called unbalanced problem.

In such cases before applying Hungarian method dummy rows or columns are added in the matrix (with zeroes as a cost element) in order to make it a square matrix.

Restrictions on Assignment

Sometimes it may so happen that a particular resource (say a man or a machine) can't be

assigned to a particular activity (or Job). In such cases the cost of performing that particular activity by a particular resource is considered to be very large (written as 'M' or '∞') so as to prohibit the entry of dispair of resource activity into the final solⁿ.

		Location machine →				
		A	B	C	D	E
machine	M ₁	9	11	15	10	11
	M ₂	12	9	10	10	9
	M ₃	10	11	14	11	7
	M ₄	14	8	12	7	8
	M ₅	0	0	0	0	0
1st		A	B	C	D	E
	M ₁	10	2	6	1	2
	M ₂	3	10	10	1	10
	M ₃	10	8	7	3	10
	M ₄	7	1	5	10	1
	M ₅	10	10	10	10	10

M₁ → A → 9
M₂ → B → 9
M₃ → E → 7
M₄ → D → 7
M₅ → C → 0
32

TRANSPORTATION PROBLEM

The structure of transportation problem involves a large no. of shipping routes from several supply centres to several demand centres.

Thus, objective is to determine shipping routes betⁿ supply centres & demand centres in order to satisfy the required quantity of goods or services at each destination centres with available quantity of goods or services at each supply centres at minimum transportation cost and/or time.

* Mathematical Model of Transportation Problem:

A Company has 3 production facilities S_1, S_2, S_3 with production capacity of 7, 9, 18 units resp. per week of a product resp. These units are to be shipped to 4 warehouses D_1, D_2, D_3, D_4 with requirement of 5, 8, 7, 14 units per week resp. The transportation cost (in rupees) per unit betⁿ factories to warehouses are given in table below:

	D_1	D_2	D_3	D_4	Supply
S_1	$9x_{11}$	$30x_{12}$	$50x_{13}$	$10x_{14}$	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

balanced q^{ty}
total supply = demand
i.e. 34 = 34

formulate this transportation problem as LP model to minimize the total cost.

Let x_{ij} = no. of units of product to be transported from facility i ($i=1,2,3$) to warehouse j ($j=1,2,3,4$).

The transportation problem is stated as LP model as follows:

$$\text{Min (Total transportation cost)} Z = 19x_{11} + 30x_{12} + 50x_{13} + 10x_{14} + 70x_{21} + 30x_{22} + 40x_{23} + 60x_{24} + 40x_{31} + 8x_{32} + 70x_{33} + 20x_{34}$$

Subject to the Constraints

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 7$$

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 9$$

$$x_{31} + x_{32} + x_{33} + x_{34} \leq 18$$

} → supply

and

$$x_{11} + x_{21} + x_{31} = 5$$

$$x_{12} + x_{22} + x_{32} = 8$$

$$x_{13} + x_{23} + x_{33} = 7$$

$$x_{14} + x_{24} + x_{34} = 14$$

} → Demand

→ In the above LP model; $x_{ij} \geq 0$ for $i=1,2,3$ and $j=1,2,3,4$.

Also there are $m \times n = 3 \times 4 = 12$ decision variables x_{ij} and $m+n=7$ constants where m is the no. of rows & n is no. of columns in the General Transportation Problem.

* General Mathematical Model of Transportation Problem

Let there be m sources of supply $S_1, S_2, \dots, S_m$ having a_i ($i=1,2,\dots,m$) having units of supply respectively to be transported to n destinations $D_1, D_2, \dots, D_n$ with b_j ($j=1,2,\dots,n$) units of demand resp.

Let c_{ij} be the cost of shipping one unit of the product from source i to destination j . If x_{ij} represents no. of units shipped from

source i to destination j , the problem is to determine the transportation schedule so as to minimize total transportation cost while satisfying supply & demand cond's.

Mathematically the transportation problem in general may be stated as follows:

$$\text{Min (Total transportation cost)} Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

subject to constraints, $\sum_{j=1}^n x_{ij} = a_i, i=1, 2, \dots, m$
for supply.

for demand, $\sum_{i=1}^m x_{ij} = b_j, j=1, 2, \dots, n$

and $x_{ij} \geq 0 \forall i, j$.

★ Existence of feasible solⁿ

A necessary & suff^y condⁿ for a feasible solⁿ to the transportation eqⁿ is:

total supply = total demand.

$$\text{i.e. } \sum_{i=1}^m a_i = \sum_{j=1}^n b_j \quad (\text{also called RIM Cond}^n)$$

	D_1	D_2	...	D_n	Supply
S_1	$(1) c_{11}$	$(2) c_{12}$		$(n) c_{1n}$	a_1
S_2	$(2) c_{21}$	$(2) c_{22}$		$(2n) c_{2n}$	a_2
...					
S_m	$(2n) c_{m1}$	$(2n) c_{m2}$		$(2n) c_{mn}$	a_m
Demand	b_1	b_2		b_n	$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$

In this problem there are $m+n$ constraints, 1 for each source of supply & destination and $m \times n$ variables.

Since all $m+n$ constraints are eq's $\therefore$ one of these eq's is extra.

The extra constraint can be derived from other constant without affecting the feasible solⁿ. It follows that any feasible solⁿ for a transport ~~eqⁿ~~ must have $m+n-1$ non-ve basic variable (or allocations) x_{ij} satisfying the RIM cond's.

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Remark ① When total supply = total demand, it is called balanced transportation problem otherwise unbalanced.

The unbalanced " " can be made balanced by adding dummy supply centre (row) or a dummy demand centre (column) as the need arises.

② When no. of the allocations (values of decision variables) at any stage of the feasible solⁿ is less than the required nos (rows + columns - 1) ($m+n-1$) i.e. no. of independent constants eq's, the solⁿ is s.t.b. Degenerate. Otherwise, non-degenerate.

③ Cell in a transportation table having the allocation i.e. $x_{ij} > 0$ are called occupied cell. Others i.e. are known non-occupied.

Q

	D_1	D_2	D_3	D_4	Supply
S_1	2 ⁽⁶⁾	3	11	7	6
S_2	1 ⁽¹⁾	0	6	1	1
S_3	5	8 ⁽⁵⁾	15 ⁽²⁾	9 ⁽²⁾	10
Demand	7	5	3	2	17

Here transportation problem is balanced
 But no. of allocation = 5
 and $m+n-1 = 6$
 i.e. both unequal.
 $\Rightarrow$ solⁿ is degenerate.

	D ₁	D ₂	D ₃	D ₄	Supply
S ₁	19 ^⑤	20 ^②	50 ^③	10 ^④	7
S ₂	70	30 ^⑥	40 ^③	60 ^④	9
S ₃	40	8	70 ^④	20 ^④	18
Demand	5	8	7	14	34

Total feasible cost = $19 \times 5 \rightarrow 95$

$30 \times 2 \rightarrow 60$

$30 \times 6 \rightarrow 180$

$40 \times 3 \rightarrow 120$

$70 \times 4 \rightarrow 280$

$20 \times 14 \rightarrow 280$

here, $m+n-1 = 3+4-1=6$

and we have 6 no. of costs. So, 1015 is self of it.

1015

LEAST COST METHOD

	D ₁	D ₂	D ₃	D ₄	Supply
S ₁	19 ^⑤	20 ^②	50 ^③	10 ^④	7
S ₂	70 ^②	30 ^⑥	40 ^③	60 ^④	9
S ₃	40 ^③	8 ^④	70 ^④	20 ^④	18
Demand	5	8	7	14	34

No. of allocations = $m+n-1$ so, Non-degenerate.

Total cost $\rightarrow$ 140

120

64

280

70

140

814

	D ₁	D ₂	D ₃	D ₄	Supply
S ₁	2 ^⑥	3 ^⑥	11	7	6
S ₂	1 ^⑥	8 ^⑥	6	1	1
S ₃	3 ^⑥	8 ^⑥	15 ^②	9 ^②	10
Demand	7	5	3	2	17

No. of allocations = $m+n-1$ so, non-degenerate.

Total cost $\rightarrow 2 \times 6 + 5 \times 1 + 0 \times 1 + 8 \times 4 + 15 \times 3 + 9 \times 2$
 $\rightarrow 12 + 5 + 32 + 45 + 18 = 112$

VOGEL APPROXIMATION METHOD

	D ₁	D ₂	D ₃	D ₄	Supply	1 st	2 nd	3 rd
40	S ₁	19 ^⑤	20 ^②	50 ^③	10 ^④	7	9	9
20	S ₂	70	30	40 ^⑦	60 ^④	9	10	20
—	S ₃	40	8 ^③	70	20 ^④	18	12	20
—	Demand	5	8	7	14	34	—	50
1 st Column diff.		21	22	10	10			
2 nd Column diff.		21	—	10	10			
3 rd Column diff.		—	—	10	10			
4 th Column diff.		—	—	10	50			

Non-degenerate problem.

Total cost $\rightarrow 19 \times 5 + 8 \times 8 + 40 \times 7 + 10 \times 2 + 60 \times 2 + 20 \times 10$
 $\rightarrow 95 + 64 + 280 + 20 + 120 + 200 = 779$

Test for Optimality

Once an initial solⁿ is obtained. The next step is to check its optimality in terms of feasibility of the solⁿ and total minimum transportation cost. The test of optimality begins by calculating an opportunity cost associated with each unoccupied cell in the transportation table.

An unoccupied cell with the largest -ve opportunity cost is selected to include in the new set of transportation allocations. This value indicates per unit cost deduction that can be achieved by making appropriate allocation in the unoccupied cell. This cell is known as an Incoming cell (or Variable).

The outgoing cell from the current solⁿ is the occupied cell (basic Variable) where allocation will become zero as allocation is made in the unoccupied cell with the largest -ve opportunity cost.

Such an exchange reduces the total transportation cost. The process is continued until no -ve opportunity cost is left. i.e. the current solⁿ is optimal solⁿ.

The modified distribution (MODI) Method also called u-v method OR Method of Multipliers is used to calculate opportunity cost associated with each unoccupied cell and then improving the current solⁿ leading to an optimal solⁿ.

* Steps of MODI Method

The steps to evaluate the unoccupied cells are as follows:-

Step 1) For an initial basic feasible solⁿ with m rows & n columns. Calculate u_i & v_j for rows & columns. The initial solⁿ can be obtained by any of the three methods done earlier.

To start with any one of u_i & v_j is assigned the value 0. It's better to assign 0 to a particular u_i or v_j where there are max. no of allocations in a row or a column resp., as this will reduce the considerable arithmetic work. The values of u_i & v_j 's for other rows & columns is calculated by using the relationship

$$C_{ij} = u_i + v_j \quad \text{for all occupied cell } (i, j)$$

Step 2) For unoccupied cell, calculate the opportunity cost by using relationship

$$d_{ij} = C_{ij} - (u_i + v_j) \quad \forall i \text{ and } j$$

Step 3) Examine sign of each d_{ij}

i) if $d_{ij} > 0$, then current basic feasible solⁿ is optimal.

ii) if $d_{ij} = 0$, then current basic feasible solⁿ will remain unaffected, but an alternate solⁿ exist.

iii) if one or a more $d_{ij} < 0$ then an improved solⁿ can be obtained by entering an unoccupied cell (i, j) into the solⁿ basis (table).

An occupied cell having the largest -ve value of d_{ij} is selected for entering into the solⁿ basis (table).

Step 4 Construct a closed ^{path} ~~box~~ (Loop) for the unoccupied cell with the largest -ve value of d_{ij} . Start the closed path with the selected unoccupied cell and mark a plus sign (+) in the cell. Trace a path along the rows or columns to an occupied cell, mark the corners with a minus sign (-) & continue down the column (row) to an occupied cell.

Then mark the corner with plus sign (+) & the minus sign (-) alternately. Close the path back to the unoccupied cell.

Step 5 Select the smallest quantity amongst the cells marked with minus sign (-) on the corners of closed loop. Allocate this value to the selected unoccupied cell & add it to occupied cells marked with plus sign (+) & subtract it from the occupied cells marked with (-) sign.

Step 6 Obtain a new improved solⁿ by allocating units to the unoccupied cell acc. to step 5 & calculate the new total transportation cost.

Step 7 Test optimality of the revised solⁿ. The procedure terminates when all $d_{ij} \geq 0$ for unoccupied cells.

* Vogel's Approximation Method

This method is preferred over NWCR & LCM methods. In this

Date: _____
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method, an allocation is made on the basis

of the opportunity (or penalty or extra) cost that would have been incurred if the allocation contains cells with min unit transp. cost are missed. Hence, the allocations are made such a way that the penalty cost is minimized. An initial solⁿ obtained by using this method is nearer to an optimal solⁿ or the optimal solⁿ itself. The steps are:-

Step 1 Calculate the penalties for each row (column) taking the difference b/w the smallest & next smallest unit transportation cost in a same row (column). This diff indicates the penalty or extra cost that has to be paid. Decision-making falls to allocate to the cell with the minimum unit transportation cost.

Step 2 Select the row or column with the largest penalty & allocate as much as possible in the cell that has the least cost in the selected row or column that satisfies the min conditions. If there is a tie in the values penalties, it can be broken by selecting the cell where the max allocation can be made.

Step 3 Adjust the supply & demand and cross it the satisfied row or column. If a row & column are satisfied simultaneously, only one of them is crossed out & the remaining row (column) is assigned a zero supply (demand).

Any row or column with zero supply or demand should not be used in computing future penalties.

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Step 4 Repeat steps 1 to 3 until the available supply at various sources & demand at various destinations is satisfied.

Remark 1. The closed loop (path) starts & ends at the selected unoccupied cell. It consists of successive horizontal & vertical (connected) lines whose end points must be occupied cells, except an end point associated with entering unoccupied cell. This means that every corner element of the loop must be an occupied cell.

It is immaterial whether the loop is traced in a clockwise or anti-clockwise dirⁿ & whether it starts up down, right or left (but never diagonally). However for a given cell only one loop can be constructed for each unoccupied cell. ∴

2. There can only be one plus (+) sign & only one (-) sign in any given row or column.

3. The closed path indicates changes involved in reallocating the shipments.

Loop in Transportation Table and its Properties :-

any basic feasible solⁿ must contain $m+n-1$ non zero allocations provided.

- (i) any two adjacent cells of the ordered set lie either in the same row or in the same column.
- (ii) no three or more adjacent cells in the ordered set lie in the same row or column. The first cell of the set must follow the last in the set, i.e. each cell (except the last) must appear only once in the ordered set.

Remarks

- (1) any loop has an even no. of cells & has atleast four cells.
- (2) The allocations are said to be independent position if it is not possible to increase or decrease any individual allocation without changing the pos^{ns} of these allocations, or if a closed loop can't be formed through these allocations without violating the sum conditions.
- (3) each row and column in the transportation table should have only one plus & minus sign. All cells that have a plus or a minus sign, except the starting unoccupied cell, must be occupied cells.
- (4) Closed loops may be or may not be in the shape of a square.

Degeneracy in the Transportation prob^{lem}
when no. of allocations in the basic feasible solⁿ is less than $m+n-1$ then the problem of degeneracy occurs.

- Degeneracy problem can occur in two ways
1. Basic feasible solⁿ may be degenerate from the initial stage onwards.
 2. Degeneracy may occur during the optimality test.

Visualisation of the degeneracy during the initial stage.

1. The extremely small quantity denoted by δ or ϵ is introduced in the least cost independent cell subject to the following assumptions. If necessary two or more deltas can be introduced in the least and second least cost independent cell.

$$(i) \delta + 0 = \delta \quad (ii) \delta - 0 = \delta$$

$$(iii) x_{ij} + \delta = x_{ij} - \delta = x_{ij}$$

If there are more than one delta in the solⁿ $\delta < \delta'$ whenever δ is ab^s.

If δ and δ' are in the same row $\delta < \delta'$ when δ is to the left of δ' .

Q1. Find the optimal solⁿ for

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opt cost = $60 \times 3 + 50 \times 3 + 20 \times 9$
= 750

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	X	Y	Z	
A	8	7	603	60
B	503	8	209	70
C	11	803	5	80
	50	80	80	

Here $\sum a_i = 210 = \sum b_j$
So, it is a balanced transportation problem.

Using Vogel's Approximation Method,
initial basic feasible solⁿ is

Here no. of allocations = 4
 $m+n-1 = 3+3-1 = 5$
 $4 < 5 (m+n-1)$

So, degeneracy occurs.

To find u_i & v_j

			u_i
		3	3
	3	9	9
		5	5
v_j	-6	-2	0

To find d_{ij}

11	6	
12	1	

Here
all $d_{ij} > 0$

current solⁿ is
⇒ A optimal solⁿ

Q2. Solve the Trans prob to find
opt solⁿ

	X	Y	Z	
I	5(2)	7	4	5
II	3	3	8(1)	8
III	5	7(4)	7	7
IV	2(1)	2(6)	10(2)	14
	7	9	18	

$\sum a_j = 34 = \sum b_j$ balanced TP.

By VAM, no. of all. = $6 = 6^{(m+n-1)}$
non degenerate.

To find u_i & v_j

				u_i
	2			1
		4		-1
	1	2	2	-2
v_j	1	6	2	0

d_{ij}

		0	1
3	-2		
6		7	

Here all d_{ij} is not +ve
current solⁿ is not
optimal.

5(2)	7	4	5
3	10(3)	8(1)	8
5	7(4)	7	7
2(1)	2(6)	10(2)	14
7	9	18	

$$\min(8-0), \min(8-0, 2-0) = 0,$$

$$2-0 = 0$$

$$\boxed{0 = 2}$$

Now New solⁿ is

	X	Y	Z	
I	5(2)	7	4	5
II	3	2(3)	6(1)	8
III	5	7(4)	7	7
IV	2(1)	6	12(2)	14
	7	9	18	

u & v

	2		1
		3	1
		4	0
1			2
	1	4	2

dij

	2	1
3		
4		5
	2	

opt cost = 768. $\checkmark$

Exercise

(i)

6(6)	4(8)	1	5	14
8	9(2)	2(4)	7	16
4	3	6(1)	2(4)	5
6	10	15	4	

(ii)

13(200)	11	15	20	2000
17(100)	14(300)	12(200)	13	6000
18	18	15(200)	12(500)	4000
3000	3000	4000	5000	

(iii)

9(3)	12(2)	9	8	4	3	5
7	3(2)	6(5)	8(1)	9	4	8
4	5	6	8(6)	10	14	6
7	3	5	7	10(6)	9(1)	7
2	3	8	10	2	4(3)	3
3	4	5	7	6	4	

(2) (i)

50	30(1)	220	1
90(3)	45(1)	110	4
250(1)	200	50(3)	7
4	2	3	

(ii)

6	3(12)	5	4(1)	22
5	9	2(15)	7	15
5	7	8	6	8
7	12	17	9	

6	3(12)	5(1)	4(1)	22
5	9	2(15)	7	15
5(1)	7	8(1)	6	8
7	12	17	9	

Q TP/20. by VAM

4②

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2⑤	7	4	5	2	2	5	-
3	3	1⑧	8	2	-	-	-
5	4②	7	7	1	1	1	1
1②	6②	2⑩	14	1	1	5	5
4	9	18					

1	1	1
1	2	2
1	2	-
4	2	-

no. of allocations = 6.
non degenerate

u _i v _j	2			1
			1	-1
		4		-2
	1	6	2	0
v _j	1	6	2	

d _{ij}		0	1
	3	-2	
	6		4

all d_{ij} are not +ve
current solⁿ is not optimal

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2⑤	7	4	5
3	+03	1⑧	8
5	4②	7	7
1②	-06②	-02⑩	14
4	9	18	

$$\min(8-0, 2-0) = 0$$

$$2-0 = 0 \Rightarrow 0 = 2$$

new solⁿ =

2⑤	7	4	5
3	3②	1⑥	8
5	4②	7	7
1②	6	2⑩	14
4	9	18	

u _i v _j	2			1
			1	-1
		4		0
	1		2	0
v _j	1	4	2	

d _{ij}		2	1
	3		
	4		5
		2	

all d_{ij} ≥ 0
current solⁿ is optimal

eg 3

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20	2	10	4	30	0	0	1	1
3	3	20	10	50	1	1	1	1
4	2	5	9	20	2	2	3	-
20	40	30	10					

2	0	1	3	no of allocation
2	0	1	-	= 6.
-	0	1	-	non-degenerate.
-	1	1	-	

u _i v _j					u _i
	1	3	2	1	9
		2			0
					-1
v _j	3	2	1		

d _{ij}		0		4	
	1				
	3		4	9	

all d_{ij} > 0

eg TP/34 Q5

27	28	31	69	150	4	4	4	4
10	45	40	72	40	22	22	-	-
30	54	35	57	80	5	5	5	5
90	70	50	60					
17	22	4	25	Min cost				
17	-	4	25	= 8190				
3	-	4	12					
3	-	4	-					

no. of allocations = 6 = 4 + 3 - 1 = 6
non degenerate

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u _i v _j		27	28	31		0	69
					32	22	54
		30			57	3	15
v _j	27	28	31	54			

d _{ij}					15		
		5	44	31			
			28	1			

all d_{ij} > 0

current optimal solⁿ

9	5	8	5	225	0	0	4
9	10	13	7	15	2	1	1
14	5	3	7	100	2	2	0
125	80	95	100				
0	0	5	2				
0	0	-	2				
0	0	-	-				

no. of allocation = 6 = 4 + 3 - 1 = 6
non-degenerate

$$\text{Min cost} = 9 \times 30 + 8 \times 95 + 5 \times 100 + 9 \times 75 + 14 \times 20 + 5 \times 80 = 2885$$

u _i v _j		9		8	5	9
		9				9
		14	5			14
v _j	0	-9	-1	-4		

	5		
	10	5	2
		-10	-3

dij < 0.
current solⁿ is not optimal

9 ⁽⁵⁰⁾	5	8 ⁽⁹⁵⁾	5 ⁽¹⁰⁰⁾	225
9 ⁽⁷⁵⁾	10	13	7	75
14 ⁽²⁰⁾	5 ⁽⁸⁰⁾	3 ⁽¹⁰⁾	7	100
125	80	95	100	

min { 20-0, 95-0 } = 0
0 = 20

9 ⁽²⁰⁾	5	8 ⁽⁷⁵⁾	5 ⁽¹⁰⁰⁾
9 ⁽⁷⁵⁾	10	13	7
14	5 ⁽⁸⁰⁾	3 ⁽¹⁰⁾	7

Q11

By VAM

3 ⁽⁶⁾	6	8 ⁽⁵⁵⁾	20	2	2	2
6	1 ⁽¹⁹⁾	2	5 ⁽⁹⁾	28	1	3
7	8	3 ⁽¹³⁾	9 ⁽⁴⁾	17	4	4
15	19	13	18			
9	5	1	0	no of allocation = 6		
3	-	1	0			
3	-	-	0			

no of allocation
= 6

non-degenerate.

$$\begin{aligned} \text{Min cost} &= 3 \times 15 + 5 \times 5 + 1 \times 19 + 5 \times 9 + 3 \times 13 + 9 \\ &= 45 + 25 + 19 + 45 + 39 + 36 \\ &= 209 \end{aligned}$$

u _i \ v _j	B		A	5	u _i
		1		5	5
			3	9	9
v _j	-2	-4	-6	0	

dij		3	9	
	3		3	
		3		

all dij > 0

Current opting solⁿ

3 ⁽¹⁵⁾	6	8	5
6	1	2	5
7	8	3	9

17

5	4.75	4.25	110000
5	5.5	6.75	20000
4.5	6	5	330000
5.5	6	4.5	440000
275000	50000	660000	→ 985.

900.

net Balanced.

eg TP-748

Q2(i)

4 ⁽¹⁾	7	3 ⁽¹⁾	8	2 ⁽²⁾	4	1	1	1	1	1
1 ⁽²⁾	4	7	3	8	7	2	2	-	-	-
7	2 ⁽³⁾	4 ⁽⁶⁾	7	2	9	2	2	2	3	-
8	8	2	4 ⁽²⁾	7	2	2	2	2	2	2
8	3	7	2	2	22					
3	2	1	1	5						
3	2	1	1	-						
3	5	1	3	-						
3	-	1	3	-						
4	-	1	4	-						

$m+n-1 = 4+5-1 = 8$
but no of allocations = 7
 $7 < 8$.

degenerate.

$$\begin{aligned} \text{min cost} &= 4 \times 1 + 1 \times 7 + 2 \times 3 + 3 \times 1 + 4 \times 6 \\ &\quad + 4 \times 2 + 2 \times 2 \\ &= 4 + 7 + 6 + 3 + 24 + 8 + 4 \\ &= 56 \end{aligned}$$

uivj

	4		3		2	-1
	1					-4
		2	4			0
				4		
Vj	5	2	4		3	

③

55	30 ⁽¹⁰⁾	40 ⁽²⁰⁾	50	40 ⁽¹⁰⁾	40	10	10
35 ⁽²⁰⁾	30	100	45	60	20	5	5
40 ⁽⁵⁾	60	95	35 ⁽³⁰⁾	30 ⁽³⁾	40	5	5
25	10	20	30	15			
5	0	55	10	10			
5	0	-	10	10			
5	0	-	-	10			
5	-	-	-	10			
15	-	-	-	10			

⑦

10 ⁽¹⁰⁾	20	5	7	10	2	5	10	10	-
13	9 ⁽¹⁰⁾	12	8	20	1	3	4	4	1
4 ⁽³⁰⁾	15	7	9	80	3	3	11	-	
14	7 ⁽¹⁰⁾	130	0 ⁽¹⁰⁾	40	1	6	7	7	
3 ⁽²⁰⁾	12 ⁽³⁰⁾	6	9	50	3	3	9	9	
60	60	20	10						
1	2	4	7						
1	2	4	-						
1	2	-	-						
7	2	-	-						
10	2	-	-						

Min cost

Transportation Problems unbalanced TP

13 Sep, 2023

Date: _____
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If in a Trans prob ~~to~~ sum of all available quantities is not equal to the sum of requirements
ie $\sum_{i=1}^n a_i \neq \sum_{j=1}^m b_j$

then such problem is called unbalanced Transportation problem.

Working Rule

whenever $\sum a_i < \sum b_j$, introduce a dummy source in the Transportable table. The cost of transportation from this dummy source to any destination are all set $= 0$. The availability at this dummy source is assumed to be equal to the difference $\sum b_j - \sum a_i$.

	D_1	D_2	D_3	a_i
O_1	4	3	2	10
O_2	2	5	0	13
O_3	3	8	6	12
b_j	8	5	4	

$\sum a_i = 35$
So, it is unbalanced TP.

	D_1	D_2	D_3	Dummy	
O_1	4	3	2	0	10
O_2	2	5	0	0	13
O_3	3	8	6	0	12
	8	5	4	18	35

Now it is balanced

4 (8)	3 (2)	0	10	2	2	2	3
2 (5)	0 (4)	0 (6)	13	0	0	0	5
3 (8)	0 (12)	0	12	3	-	-	-
8	5	4	18				

1	2	2	0
2	2	2	0
-	2	2	-
-	2	-	0

no. of allocation

$$= 6 = 3 + 4 - 1$$

non deg solⁿ

$$\text{min cost} = 4 \times 8 + 3 \times 2 + 5 \times 3 + 0 \times 4 + 0 \times 6 + 12$$

$$= 32 + 6 + 15 = 53$$

$u_i \& v_j$	4	3			u_i
		5	0	0	-2
				0	0
v_j	6	5	0	0	

d_{ij}			4	2	
	-4				
	-3	3	6		

next optimal

4 (8)	3 (2)	0			
2 (5)	0 (4)	0 (6)			
3 (8)	0 (12)	0			

$$\min \{ 8-0, 3-0 \}$$

$$0 = 3$$

4 (5)	3 (5)	2	0		
2 (3)	5	0 (4)	0 (6)		
3	8	6	0 (12)		

$$20 + 15 + 6 = 41$$

u_i & v_j

					u _i
	4	3			2
	2		0	0	0
				0	0
v _j	2	3	0	0	

d_{ij}

			0	-2
		4		
	1	7	6	

Not Optimal

4	3	2	0
2	5	0	0
3	8	6	0

Min {5-0, 6-0}
0 = 5

4	3	2	0
2	5	0	0
3	8	6	0

15 + 16 = 31 Opt sol

u_i & v_j

					u _i
		3	0	0	0
	2		0	0	0
			0	0	0
v _j	2	3	0	0	

d_{ij}

	2		2
		2	
	1	5	6

all
d_{ij} ≥ 0

Min opt solⁿ = 31

Maximisation Transportation Problem

step 1) convert the given problem into minimization by replacing each element of the transportation table its difference from the maximum element of the table.

					a_i
$\downarrow$	20	21	16	18	10
	17	28	14	16	9
	29	23	19	20	7
b_j	6	10	4	5	

z_{ai} = 26, z_{bj} = 25
unbalanced

20	21	16	18	0	10	16	2	2	-
17	28	14	16	0	9	14	2	2	2
29	23	19	20	0	7	19	1	1	1
6	10	4	5	1	26				

3	2	2	2	0
3	2	2	2	-
-	2	2	2	-
-	-	5	4	-

no. of all. = 6 < m+n-1 = 3+5-1 = 7
degenerate

solⁿ 21x10 + 17x6 + 14x3 + 9x1 + 20x5
= 210 + 463

Test for optimality

solⁿ

27/Sept/23.

UNIT-3

6

DATE

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GAME THEORY

Defⁿ: ① Game is defined as an activity b/w two or more persons involving activities by each person according to a set of rules at the end of which each person receives some benefit or satisfaction or negative benefit.

② The set of rules defines the game. Going through the set of rules once by participants defines a play.

Characteristics of Game Theory

① Chance of Strategy : If in a game activities are determined by skill, it is said to be a game of strategy. If they are determined by chance, then it is the game of chance. In general, a game may involve game of strategy as well as a game of chance.

② No. of Persons : A game is called an 'n' person game if the no. of persons playing is 'n'. The person means an individual or a group aiming at a particular objective.

③ No. of Activities : These may be finite or infinite.

④ No. of alternatives (choices) available to each person: In a particular activity, no. of choices may be finite or infinite. A finite game has a finite no. of activities each involving a finite no. of choices, otherwise a game is said to be infinite.

⑤ Information to the players about the past activities of other players is completely available, partially available or not available at all.

⑥ Pay-off: A quantity measure of satisfaction, a person gets at the end of each play is called a payoff. It is a real valued function of the variables in a game.

Let v_i be the payoff to the player P_i , $1 \leq i \leq n$ in an 'n' person game.

If $\sum_{i=1}^n v_i = 0$, then the game is said to

be a zero sum game.

Some - Basic Definitions

1. Competitive Game: Competitive games are those in which players play against one another and where one player winning means another player loses. The games like chess, poker etc. have the characteristics

of competition and are played according to some definite rules. Game theory provides optimal solutions to such games, assuming that each of the players wants to maximize his profit or minimize his loss.

OR

A competitive situation is called a competitive game if it has the following four properties:

(i) There are finite no. (n) of competitors (called players) s.t. $n \geq 2$. In case, $n=2$, it is called a two-person game and in case $n > 2$, it is referred to as a n -person game.

(ii) Each player has a list of finite no. of possible activities.

(iii) A play is said to occur when each player chooses one of his activities. The choices are assumed to be made simultaneously i.e., no player knows the choice of the other until he has decided on his own.

(iv) Every combination of activities determines an outcome which results in a gain of payments to each player, provided each player is playing uncompromisingly to get as much as possible. -ve gain implies the loss of same amount.

2. Zero sum and non-zero sum Game :
If the players make payments only to each other i.e. the loss of one is the gain of other, and nothing comes from outside, the competitive game is said to be zero sum game.

A game which is not zero-sum is called a non-zero sum game.

3. Strategy : A strategy of a player has been loosely defined as a rule for decision-making in advance of all the plays by which he decides the activities he should adopt.

In other words, a strategy for a given player is the set of rules which specifies which course of the available course of action he should make at each play.

The strategy may be of two kinds :

(i) Pure strategy :- The pure strategy is a decision rule always to select a particular course of action.

(ii) Mixed-strategy :- The mixed strategy is a selection among pure strategies with fixed probabilities.

4. Two-person zero sum (or Rectangular) Games :
A game with only two players is called a two-person zero sum game. If the losses of one player are equivalent of the other, so that the sum of their net gains is zero.

5. Pay-off Matrix : Suppose the player A has 'm' activities and player B has 'n' activities, then a pay-off matrix can be formed by adopting the following rules :

(i) Row designations for each matrix are activities available to player A.

(ii) Column designations for each matrix are activities available to player B.

(iii) Cell entry V_{ij} is the payment to the player A in A's payoff matrix when A select the activity i & B choose the activity j.

(iv) With a zero-sum two person game, the cell entry in the player B payoff matrix will be negative of the corresponding cell entry V_{ij} in the player A payoff matrix. So that sum of the payoff matrices for player A & player B is zero.

		Player B					
		1	2	...	j	...	n
Player A	1	V_{11}	V_{12}	...	V_{1j}	...	V_{1n}
	2	V_{21}	V_{22}	...	V_{2j}	...	V_{2n}
	...						
	i	V_{i1}	V_{i2}	...	V_{ij}	...	V_{in}
	m	V_{m1}	V_{m2}	...	V_{mj}	...	V_{mn}

A's payoff matrix

		Player B					
		1	2	...	j	...	n
Player A	1	$-V_{11}$	$-V_{12}$	...	$-V_{1j}$	...	$-V_{1n}$
	2	$-V_{21}$	$-V_{22}$	...	$-V_{2j}$	...	$-V_{2n}$
	...						
	i	$-V_{i1}$	$-V_{i2}$	...	$-V_{ij}$	...	$-V_{in}$
	m	$-V_{m1}$	$-V_{m2}$	...	$-V_{mj}$	...	$-V_{mn}$

B's payoff matrix

28/Sept/23

Maximin Principle: For a player A, the minimum value in each row represents the least gain (pay off) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that gives the largest gain among the row minima value. This choice of player A is called ~~maximin~~ maximin principle and the corresponding gain is called the maximin value of the game.

Minimax Principle: For player B who is assumed to be the loser, the maximum value in each column the maximum loss to him, if he chooses this particular strategy. These are written in the payoff matrix by column maxima. He will select the strategy

that give min. loss among the column max. value. The choice of player B is called minimax principle and the corresponding value (loss) is called the minimax value of the game.

Optimal Strategy: If the maximin value is equal to minimax value, then the game is said to have a saddle (equilibrium) point and the corresponding strategy is called optimal strategy.

Value of the game: This is the expected pay off at the end of the game when each player uses his optimal strategy i.e. the amount of payoff matrix at the equilibrium point. A game may have more than one saddle point. A game with no saddle point is solved by choosing strategies with fixed probabilities.

Remark: ① The value of the game satisfy the equation $\text{Maximin value} \leq V \leq \text{minimax value}$.

② A game is said to be a fair game if the lower (maximin) and upper (minimax) value of the game are equal and both equal to zero.

③ A game is said to be strictly determinable if the lower (maximin) and upper (minimax) value of the game are equal & both equal

to the value of the game.

Rules to determine saddle point

- ① Select the lowest element in each row of the payoff and write them under row minima heading, then select the largest element among these elements and enclose it in a rectangle.
- ② Select the largest element in each column of the payoff matrix and write them under column maxima heading. Then select the lowest element among these element and enclose it in a circle.
- ③ Find out the element i.e., same in the circle as well as rectangle and mark the position of such element in the matrix. This element represent the value of the game and it is called the saddle point or equilibrium point.

Ques

	Player B			
Player A	B ₁	B ₂	B ₃	Row Minimum
A ₁	-1	2	-2	-2
A ₂	6	4	-6	-6
Column Maximum	6	4	-2	

Saddle point.

3/10/23

Minimax Criterion of Optimality

It states that, if a player list the worst possible outcomes of all his potential strategies, he will choose that strategy to be the most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an optimal strategy.

Ques Consider two-person zero sum game matrix, which represents payoff to the player A. Find the optimal strategy (if any).

		B		
		I	II	III
A	I	-3	-2	6
	II	2	0	2
	III	5	-2	4

Sol: Find saddle point.

		B			Row Min.
		I	II	III	
A	I	-3	-2	6	-3
	II	2	0	2	0
	III	5	-2	4	-2
Col max		5	0	6	

maximum

minimum

Max Mini = 0 = MiniMax.

Saddle point exist here.

optimal strategy for Player A = II
optimal strategy for Player B = II

Ques Consider the following game

		B			Row Min
		I	2	3	
A	1	3	-4	8	-4
	2	-8	5	-6	-8
	3	6	-7	6	-7
Col Max		6	5	8	

5 minimum

-4 Maximum

$\text{Max mini} = -4$ mini P max
 $\text{Mini max} = 5$ max P min (no in diff)

Ques Find the range of values of p & q which will declare the entry (2,2) a saddle point for the following table.

		Player B			
		1	2	3	
Player A	1	2	4	5	2
	2	10	7	q	7
	3	4	p	6	4
		10	7	q	

$q \geq 7$ & $p \leq 7$

Ques The payoff matrix of a game is given. Find the solution of the game to the player A and player B.

		B					Row min
		I	II	III	IV	V	
A	I	-2	0	0	5	3	-2
	II	3	2	1	2	2	1
	III	-4	-3	0	-2	6	-4
	IV	5	3	-4	2	-6	-6
Col max		5	3	1	5	6	

The best strategy for player A is II & The best strategy of the player B is III.

The value of the game is 2 to Player A & -1 to player B.

Practice Work

Ques Find the saddle point and hence solve the following games:

		B			Row min
		B ₁	B ₂	B ₃	
A	A ₁	15	2	3	2
	A ₂	6	5	7	5 ← max.
	A ₃	-1	4	0	-1
Col Max		15	5	7	
			↑ min		

$\text{Max mini} = 5 = \text{Minimax} \Rightarrow \text{Saddle point exist}$
 optimal strategy for A & B = (A₂, B₂)

Ques 2

		B				Row min
		B ₁	B ₂	B ₃	B ₄	
A	A ₁	1	7	3	4	1
	A ₂	5	6	4	5	4 ←
	A ₃	7	2	0	3	0
Col max		7	7	4	5	

Minimax = 4 = Maxmini
⇒ saddle point exist

Optimal strategy for player A = A₂
Optimal strategy for player B = B₃

Ques 3

		B				Row min
		I	II	III	IV	
A	I	-5	2	1	20	-5
	II	5	5	4	6	4 ←
	III	4	-2	0	-5	-5
Col max		5	5	4	20	

Minimax = 4 = maxmini
⇒ saddle point exist

Optimal strategy for player A = II
Optimal strategy for player B = III

Ques 4

		B					Row min
		I	II	III	IV	V	
A	I	9	3	1	8	0	0
	II	6	5	4	6	7	4 ←
	III	2	4	4	3	8	2
	IV	5	6	2	2	1	1
Col max		9	6	4	8	8	

Maxmini = 4 = minmax
⇒ saddle point exist

Optimal strategy for player A = II
Optimal strategy for player B = III

Ques 5

		B			Row min
		I	II	III	
A	I	6	8	6	6 ←
	II	4	12	2	2
Col max		6	12	6	

Maxmini = 6 = Minimax

There exist two saddle points

Optimal strategies are (I, I) & (I, III)

Ques 6

		B			Row min
		C ₁	C ₂	C ₃	
A	R ₁	3	0	-3	-3
	R ₂	2	3	1	1 ←
	R ₃	-4	2	-1	-4
Col max		3	3	1	

Maxmini = 1 = Minimax
⇒ saddle point exist

Optimal strategy for player A = R₂
Optimal strategy for player B = C₂

Ques 7

Solve the games whose payoff matrices are given by.

(a)

	Player B			Row min
Player A	I	II	III	
I	1	2	1	1 ←
II	0	-4	-1	-4
III	1	3	-2	-2
Col max	1	3	1	

$\uparrow$ $\uparrow$

optimal strategies are (I, I) or (I, III)

maximin $V = 1$ for A
 $V = -1$ for B.

(b)

	B					Row min
A	I	II	III	IV	V	
I	7	8	4	6	8	4 ←
II	-8	6	1	9	6	-8
Col max	7	8	4	9	8	

$\uparrow$ $\uparrow$ $\uparrow$

optimal strategies are (I, II) or (II, I)

Minimax = 4 = Maximin.

Optimal strategy for A = II
 " " " B = I

value of game for A = 4
 value of game for B = -4.

Ques 8

Solve the games whose payoff matrices are given below:

(a)

		B			Row min
		B ₁	B ₂	B ₃	
A	A ₁	-3	-1	6	-3
	A ₂	2	0	2	0 ←
	A ₃	5	-2	-4	-4
	Col max	5	0	6	

$\uparrow$

Maximin = 0 = Minimax.
 optimal strategy = (A₂, B₂)
 $V = 0$

(b)

		B			Row min
		B ₁	B ₂	B ₃	
A	A ₁	15	2	3	2
	A ₂	6	5	7	5 ←
	A ₃	-7	4	0	-7
	Col max	15	5	7	

$\uparrow$

maximin = 5 = minimax
 optimal strategy = (A₂, B₂)
 $V = 5$

Ques 9

For what value of 'a' the game with the following pay-off matrix is strictly determinate?

		B			Row min
		B ₁	B ₂	B ₃	
A	A ₁	a	6	2	a
	A ₂	-1	a	-7	-7
	A ₃	-2	4	a	-2
	Col Max	a	6	2	

$\underbrace{\hspace{10em}}_a$

$-1 \leq a \leq 2$

2x2 Game without Saddle Point

Consider a payoff matrix of player A.

		B	
		y_1	y_2
A	x_1	V_{11}	V_{12}
	x_2	V_{21}	V_{22}

Value of the Game to the player A is given by,

$$V = \frac{V_{11} \cdot V_{22} - V_{12} \cdot V_{21}}{V_{11} + V_{22} - (V_{12} + V_{21})}$$

& the formula for the optimal strategies can be found by using the results

$$\frac{x_1}{x_2} = \frac{V_{22} - V_{21}}{V_{11} - V_{12}}$$

$$\frac{y_1}{y_2} = \frac{V_{22} - V_{12}}{V_{11} - V_{21}}$$

$$x_1 = \frac{V_{22} - V_{21}}{V_{11} + V_{22} - (V_{12} + V_{21})}$$

$$x_2 = \frac{V_{11} - V_{12}}{V_{11} + V_{22} - (V_{12} + V_{21})}$$

$$y_1 = \frac{V_{22} - V_{12}}{V_{11} + V_{22} - (V_{12} + V_{21})}$$

$$y_2 = \frac{V_{11} - V_{21}}{V_{11} + V_{22} - (V_{12} + V_{21})}$$

Ques Find out the value of the game and optimal strategy:

		B		
		y_1	y_2	Row min
A	x_1	8	-3	-3
	x_2	-3	1	-3
		col max	8	1

$\therefore$ There does not exist saddle point.

$$V_{11} = 8, V_{12} = -3, V_{21} = -3, V_{22} = 1$$

$$V = \frac{V_{11} \cdot V_{22} - V_{12} \cdot V_{21}}{V_{11} + V_{22} - (V_{12} + V_{21})} = \frac{(8 \times 1) - (-3 \times -3)}{(8 + 1) + (-3 + 3)} = \frac{8 - 9}{9 + 6} = \frac{-1}{15}$$

$$V = \frac{8 - 9}{9 + 6} = \frac{-1}{15}$$

$$x_1 = \frac{V_{22} - V_{21}}{V_{11} + V_{22} - (V_{12} + V_{21})} = \frac{1 + 3}{15} = \frac{4}{15}$$

$$x_2 = \frac{V_{11} - V_{12}}{V_{11} + V_{22} - (V_{12} + V_{21})} = \frac{8 + 3}{15} = \frac{11}{15}$$

$$y_1 = \frac{V_{22} - V_{12}}{V_{11} + V_{22} - (V_{12} + V_{21})} = \frac{1 + 3}{15} = \frac{4}{15}$$

$$y_2 = \frac{V_{11} - V_{21}}{V_{11} + V_{22} - (V_{12} + V_{21})} = \frac{8 + 3}{15} = \frac{11}{15}$$

Arithmetic method for 2x2 Games

Step 1: Find the difference of two numbers in column 1 and put it under the column 2, neglecting the negative sign, if occurs.

Step 2: Find the difference of two numbers in column 2 and put it under column 1, neglecting the negative sign, if occurs.

Step 3: Repeat the above two steps for the two rows also.

The value thus obtained are called the ~~oddments~~ oddments. These are the frequencies with which the players must use their courses of action in this optimum strategy. The optimum strategy can be found by dividing sum of the differences.

Ques.

	Player B	
Player A	I	II
	8 -3	-3 1

Find out the value of game and optimal strategy.

Solⁿ:

	Player B		Row min
Player A	I	II	
	8 -3	-3 1	-3 1
	-3 1	8 -3	-3 1
Col max	8	1	

	Player B		Row min
Player A	I	II	
	8 -3	4 11	4 11
	-3 1	4 11	4 11
	4 11	8 -3	4 11
Col max	8	11	

Optimum strategy for player A is $\left(\frac{4}{15}, \frac{11}{15}\right)$

and for player B is $\left(\frac{4}{15}, \frac{11}{15}\right)$

value of the game using player B oddments

$$V = \frac{(4 \times 8) + (11 \times -3)}{11 + 4} = \frac{32 - 33}{15} = -\frac{1}{15}$$

Ques.

Find out the value of game and optimal strategy.

	Player B		Row min
Player A	I	II	
	-4 6	2 -3	-4 2
	2 -3	-4 6	2 -3
Col max	2	6	

Minimax $\neq$ Maximin

$\therefore$ Saddle point does not exist.

	Player B		Row min
Player A	I	II	
	-4 6	2 -3	-4 2
	2 -3	-4 6	2 -3
Col max	2	6	

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optimal strategy for player A is $\left(\frac{5}{15}, \frac{10}{15}\right)$

& player B is $\left(\frac{9}{15}, \frac{6}{15}\right)$

value of the game is given by -

$$V = \frac{(5 \times -4) + (10 \times 2)}{15} = \frac{-20 + 20}{15 + 15} = 0$$

$$V = 0$$

It means the game is fair.

Note: If the sum of row = sum of column, then, value of game is same $\forall$.

Dominance Rules

Rule 1: If each element in one row, say r^{th} of the payoff matrix $[v_{ij}]$, is less than or equal to the corresponding element in the other row, say s^{th} , then the player A will never choose the r^{th} strategy. In other words, if $\forall j = 1, 2, \dots, n$ and $v_{rj} \leq v_{sj}$, then the probability x_r of choosing r^{th} strategy will be zero. The value of the game and the non-zero choices of probabilities remain unaltered even if r^{th} row is deleted from the payoff matrix. Such r^{th} row is said to be dominated by the s^{th} row.

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Rule 2: Following the similar arguments, if each element in one column, say C_r , is greater than or equal to the corresponding element in the other column, say C_s , then the player B will never use the strategy corresponding to column C_r . In this case, the column C_s dominates the column C_r .

Rule 3: Dominance need not be based on the superiority of pure strategies only. A given strategy can be dominated if it is inferior to an average of two or more other pure strategies. In general, if some convex linear combination of some rows dominates the i^{th} row, then the i^{th} row will be deleted. If the i^{th} row dominates the convex linear combination of some other rows, then one of the rows involving in the combination may be deleted. Similar arguments follow for columns also.

Rule 4: If (x_1, x_2) be the optimal strategy for the player A for the reduced game and (w_1, w_2) be the optimal strategy for the original game, then w_1 is the i^{th} place extension of x_1 .

Rule 5: If (y_1, y_2) be the optimal strategy for the player B for the reduced game and (y_1, w_2) be the optimal strategy for the original game, then w_2 is the j^{th} place extension of y_2 .

Rule 6: If the dominance holds strictly, then values of optimal strategies do coincide, and when the dominance does not hold strictly, then optimal strategies may not coincide.

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Ques

		Player B			Row min	
		I	II	III		
Player A	I	-4	6	3	-4	} Max -3
	II	-3	-3	4	-3	
	III	2	-3	4	-3	
Col max		2	6	4		
		Min -2				

Ans

Minimax $\neq$ Maxmini
 $\therefore$ No Saddle point

		Player B		
		I	II	
Player A	I	-4	6	On comparing column I & II, elements of II are higher. So, neglect column I.
	II	-3	-3	
	III	2	-3	

		Player B		
		I	II	
Player A	I	-4	6	On comparing, now II and III, elements of II are smaller. So, neglect II now.
	III	2	-3	

		Player B		
		I	II	
Player A	I	-4	6	5/15
	III	2	-3	10/15

Row $\rightarrow$ kum wala delete

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optimal strategy of Player A $\left(\frac{1}{3}, 0, \frac{2}{3}\right)$

optimal strategy of Player B $\left(\frac{3}{5}, \frac{2}{5}, 0\right)$

$$\text{value of game} = \frac{V_{11} V_{22} - V_{12} V_{21}}{V_{11} + V_{22} - (V_{12} + V_{21})} = \frac{12 - 12}{-15} = 0$$

Ques

		Player B						Row min
		I	II	III	IV	V	VI	
Player A	I	0	0	0	0	0	0	0
	II	4	2	0	2	1	1	0
	III	4	3	1	3	2	2	1
	IV	4	3	7	-5	1	2	-5
	V	4	3	4	-1	2	2	-1
	VI	4	3	3	-2	2	2	-2
Col max		4	3	7	3	2	2	
		2						

Maxmini $\neq$ Minimax

$\therefore$ No Saddle point.

Row I & II both will be dominated by Row III.

		Player B					
		I	II	III	IV	V	VI
Player A	III	4	3	1	3	2	2
	IV	4	3	7	-5	1	2
	V	4	3	4	-1	2	2
	VI	4	3	3	-2	2	2

Colⁿ & V dominated by Column I, II & VI

		Player B		
		III	IV	V
Player A	III	1	3	2
	IV	7	-5	1
	V	4	-1	2
	VI	3	-2	2

Row VI is dominated by V

		Player B		
		III	IV	V
Player A	III	1	3	2
	IV	7	-5	1
	V	4	-1	2

Average of Player B strategies III & IV gives
 $\left\{ 2, 1, \frac{3}{2} \right\}$ which is superior

to the Player B's Vth pure strategy.

∴ delete the Vth strategy from the matrix.

		Player B	
		III	IV
Player A	III	1	3
	IV	7	-5
	V	4	-1

Average of Player A strategies III & IV gives
 $\left\{ 4, -1 \right\}$ which is same as Vth strategy

∴ delete Row V from the matrix.

		Player B	
		III	IV
Player A	III	1	3
	IV	7	-5

$\frac{12}{14} = \frac{6}{7}$
 $\frac{2}{14} = \frac{1}{7}$
 $\frac{8}{14}$
 $\frac{6}{14}$
 $\frac{4}{7}$
 $\frac{3}{7}$

optimal strategy for player B is $\left(0, 0, \frac{4}{7}, \frac{3}{7}, 0, 0 \right)$

and for player A is $\left(0, 0, \frac{6}{7}, \frac{1}{7}, 0, 0 \right)$

$$\begin{aligned} \text{Value of game} &= \frac{V_{11}V_{22} - V_{12}V_{21}}{V_{11} + V_{22} - (V_{12} + V_{21})} \\ &= \frac{-5 - 21}{1 - 5 - 10} = \frac{-26}{-14} = \frac{13}{7} \end{aligned}$$

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Ques

		Player B				
		B ₁	B ₂	B ₃	B ₄	B ₅
Player A	A ₁	4	4	2	-4	-6
	A ₂	8	6	8	-4	0
	A ₃	10	2	4	10	12
Col max.		10	6	8	10	12

Row min: -6, -4, 2
 2 max
 6 → min

Minimax ≠ Maxmini

∴ No Saddle point.

Row I is dominated by Row III

		Player B				
		B ₁	B ₂	B ₃	B ₄	B ₅
Player A	A ₁	8	6	8	-4	0
	A ₃	10	2	4	10	12

B₁, B₂ & B₃ are dominantes B₂

		Player B		
		B ₂	B ₄	B ₅
Player A	A ₂	6	-4	0
	A ₃	2	10	12

B₅ dominates B₄

		Player B		
		B ₂	B ₄	
Player A	A ₂	6	-4	$\frac{8}{18} = \frac{4}{9}$
	A ₃	2	10	$\frac{10}{18} = \frac{5}{9}$
		$\frac{14}{18}$	$\frac{4}{18}$	
		$= \frac{7}{9}$	$= \frac{2}{9}$	

optimal strategy for Player A is $(0, \frac{4}{9}, \frac{5}{9}, 0, 0)$

& Player B = $(0, \frac{7}{9}, 0, \frac{2}{9}, 0)$

value of game = $\frac{V_{11}V_{22} - V_{12}V_{21}}{V_{11} + V_{22} - (V_{12} + V_{21})} = \frac{60 + 8}{18} = \frac{68}{18} = \frac{34}{9}$

Graphical Method [for (2xn) and (mx2) games]

Consider the (2xn) game assuming that the game doesn't have a saddle point

		Player B				
		B ₁	B ₂	...	...	B _n
		y ₁	y ₂	...	...	y _n
Player A	A ₁	x ₁	V ₁₁	V ₁₂	...	V _{1n}
	A ₂	1-x ₁	V ₂₁	V ₂₂	...	V _{2n}

Since the player A has two strategies, it follows that $x_2 = 1 - x_1$, where $x_1 \geq 0, x_2 \geq 0$.

∴ This shows that, player A's expected payoff varies linearly with x_1

B's Pure Strategies	A's expected Payoff
B ₁	$V_{11}x_1 + V_{21}(1-x_1)$
B ₂	$V_{12}x_1 + V_{22}(1-x_1)$
⋮	⋮
B _n	$V_{1n}x_1 + V_{2n}(1-x_1)$

According to the maxmini criteria for mixed strategy games. The player A should select the value of x_1 so as to maximize his minimum expected payoff. This may be done by plotting the straight lines

$$E(x_1) = (V_{11} - V_{21})x_1 + V_{21}$$

$$E(x_2) = (V_{12} - V_{22})x_1 + V_{22}$$

$$E(x_n) = (V_{1n} - V_{2n})x_1 + V_{2n}$$

The lowest boundaries of these lines will give the minimum expected payoff as function of x_1 .

The Highest point on this lowest boundary would then give the maximum expected payoff and the optimum value of x_1 is x_1^* .

Ques Solve the 2×3 game graphically.

		Player B		
		I	II	III
Player A	I	1	3	4
	II	8	5	2

Soln:

$$E(x_1) = -7x_1 + 8$$

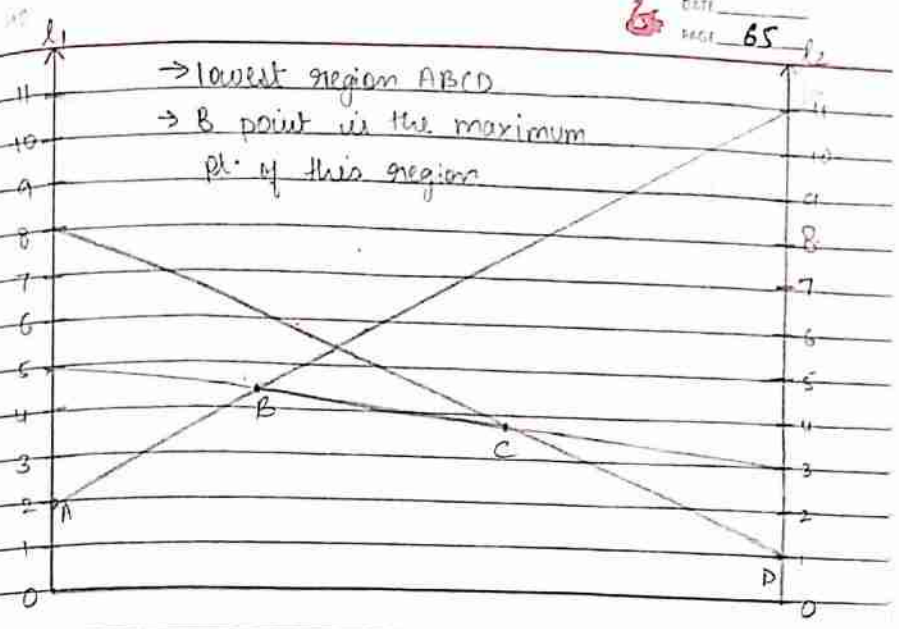
$$E(x_2) = -2x_1 + 5$$

$$E(x_3) = 9x_1 + 2$$

		B		
		II	III	
A	I	3	11	$\frac{3}{11}$
	II	5	2	$\frac{8}{11}$
		$\frac{9}{11}$	$\frac{2}{11}$	

$A \rightarrow \left(\frac{3}{11}, \frac{8}{11} \right)$
 $B \rightarrow \left(0, \frac{9}{11}, \frac{2}{11} \right)$

Value of game $\Rightarrow \frac{6-55}{5-16} = \frac{49}{11}$



Ques Solve the game graphically.

		Player B	
		I	II
Player A	I	2	4
	II	2	3
	III	3	2
	IV	-2	6

$$E(y_1) = 2y_1 + (1-y_1)4 = 2y_1 + 4 - 4y_1 = 4 - 2y_1$$

$$E(y_2) = 2y_1 + (1-y_1)3 = 2y_1 + 3 - 3y_1 = 3 - y_1$$

$$E(y_3) = 3y_1 + (1-y_1)2 = 3y_1 + 2 - 2y_1 = 2 + y_1$$

$$E(y_4) = -2y_1 + (1-y_1)6 = -2y_1 + 6 - 6y_1 = 6 - 8y_1$$

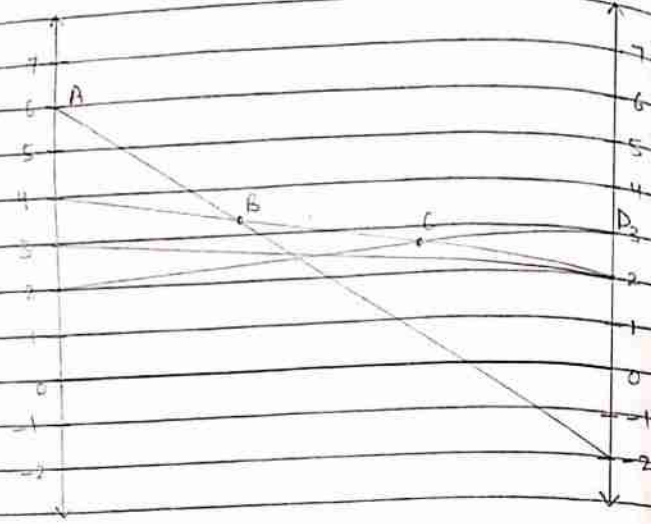
		B		
		I	II	
A	I	2	4	$\frac{1}{3}$
	III	3	2	$\frac{2}{3}$
		$\frac{2}{3}$	$\frac{1}{3}$	

$A \rightarrow \left(\frac{1}{3}, 0, \frac{2}{3}, 0 \right)$
 $B \rightarrow \left(\frac{2}{3}, \frac{1}{3} \right)$

Value of game $= \frac{4-12}{4-7} = \frac{8}{3}$

Sequencing

Suppose there are



Sequencing

Suppose there are n jobs $(1, 2, 3, \dots, n)$ each of which has to be processed one at a time at each of m machines $[A, B, C, \dots]$. The order of processing each job through machines is given [For eg:- Job 1 is processed through the machine $A, C, \dots$ in this order].

The time ~~the~~ each job must require on each machine is known. The problem is to find a sequence among $(n!)^m$ number of all possible sequences for processing the jobs so that total elapsed time T the jobs will be minimum.

Mathematically,

$$\begin{aligned} \text{Let } A_i &= T_{\text{time}} \text{ for Job } i \text{ on machine } A \\ B_i &= T_{\text{time}} \text{ for Job } i \text{ on machine } B \end{aligned}$$

$T = T_{\text{time}}$ for start of 1st Job to the completion of last job.

Then the problem is to determine for each machine a sequence of jobs $(i_1, i_2, \dots, i_n)$ where $i_1, i_2, \dots, i_n$ is the permutation of integers which will minimize T .

Ideal time on a machine: This is the time for which a machine remains ideal during the total elapsed time. The notation $[X_{ij}]$ shall be used to denote the ideal time of machine j b/w the end of the $(i-1)^{\text{th}}$ job & the start of the i^{th} job.

Total elapsed time: This is the time b/w starting the 1st job & completing the last job. This includes the ideal time if exist. It is denoted by T .

No passing rule: This rule means that the passing is not allowed i.e., the same order of Job is maintained over each machine.

If each of the n jobs to be processed through 2 machines A & B in this order AB

then this rule means that each job will go to the machine A first and then to B.

Processing 'n' jobs through 2 machines

- (i) The problem can be described as only 2 machine A & B are involved.
- (ii) Each job is processed in the order AB (say).
- (iii) The exact or expected processing time $A_1, A_2, \dots, A_n$ and $B_1, B_2, \dots, B_n$ are known.

	Job(i)					
	1	2	3	-	-	- i - - - n
Machine A	A_1	A_2	A_3	-	-	A_i - - - A_n
Machine B	B_1	B_2	B_3	-	-	B_i - - - B_n

Johnson's Algorithm

Step 1: Examine the A_i 's & B_i 's for $i=1, 2, \dots, n$ & find out minimum of $[A_i, B_i] \neq i$.

Step 2: (i) If this min is A_k for $i=k$, do the k^{th} job first of all.
(ii) If this min is B_r for $i=r$, do the r^{th} job last of all.

Step 3: (i) If there is a tie for minima i.e. $A_k = B_r$ process the k^{th} Job first & r^{th} job in the last.

(ii) If the tie for the minimum occurs among A_i 's, select the job corresponding to the min. B_i 's & process it first of all. ^[B_i max value]

(iii) If the tie for minimum occurs among the B_i 's, select the job corresponding to the min. A_i 's & process it in the last. ^[A_i min value]

Step 4: Cross out the jobs already assigned and repeat step 1 to 3 arranging the jobs next to first or next to last until all the jobs have been assigned.

Ques: There are 5 jobs each of which must go through the 2 machines AB. Processing time are given in the table: (Processing time in hours).

	Jobs				
	1	2	3	4	5
Machine A	5	1	9	3	10
Machine B	2	6	7	8	4

Determine a sequence for 5 jobs that will minimize the elapsed time T . Also Calculate the total ideal time for the machine in this period.

Solⁿ: It is seen that the smallest processing time is 1 for job 2 at machine A so,

2 | 4 | 3 | 5 | 1 ← optimal sequence.

Further we will calculate the elapsed time corresponding to min. using individual processing time.

Optimal job sequence	Machine A		Machine B	
	Time IN	Time OUT	Time IN	Time OUT
2	0	1	1	7
4	1	4	7	15
3	4	13	15	22
5	13	23	23	27
1	23	28	28	30

Ideal time for A = $30 - 28 = 2$ hr
Ideal time for B = $1 + 1 + (23 - 22) = 3$ hr

Min. elapsed time = 30 hr.

Ques In a factory there are 6 jobs to perform each of which should go through 2 machines in the order AB. The processing times (in hrs) for the jobs are given here, you have to find the sequence for performing the jobs that would minimize the total elapsed time 'T'. Also find the value of T.

Jobs	1	2	3	4	5	6
Machine A	1	3	8	5	6	3
Machine B	5	6	3	2	2	10

Solⁿ Optimal sequence :-

1	6	2	3	4	5
J ₁	J ₆	J ₂	J ₃	J ₄	J ₅

Minimum elapsed time :-

Optimum Sequence	Machine A		Machine B	
	Time IN	Time OUT	Time IN	Time OUT
1	0	1	1	6
6	1	4	6	16
2	4	7	16	22
3	7	15	22	25
4	15	20	25	27
5	20	26	27	29

Min. Total Time elapsed = 29 hr.

Ideal time :-

for machine A = $29 - 26 = 3$ hr
for machine B = 1 hr

Books	1	2	3	4	5	6
Machine A	30	120	50	20	90	100
Machine B	80	100	90	60	30	10

Solⁿ Optimal sequence :-

4	1	3	2	5	6
J ₄	J ₁	J ₃	J ₂	J ₅	J ₆

Optimum Sequence	Machine A		Machine B	
	Time IN	Time OUT	Time IN	Time OUT
4	0	20	20	80
1	20	50	80	160
3	50	100	160	250
2	100	220	250	350
5	220	310	350	380
6	310	410	410	420

Total time elapsed (T) = 420 hr

Ideal time : 2

for machine A = $(420 - 410) = 10$ hr

for machine B = $(30 + 20) = 50$ hr

- # Player: A competitor in the game is known as player.
- # Strategy: A set of alternative courses of action available to player based on advanced knowledge is known as strategy.
- # Pure Strategy: If the player selects same strategy each time,
- # Mixed strategy: If the player selects his course of action in accordance with fixed probability.
- # Optimum Strategy: A course of action which puts the player in the most preferred position.
- # Two-person zero-sum game: When only two players are involved in the game and if the gain made by some player is equal to loss of the other.
- # Fair Game: If the value of the game is zero.
- # Solution of the Game having saddle point (Pure strategy)
If the max-min value equals to Min-max value, then the game is said to have saddle point & corresponding strategy is called optimum strategy.

Ques Solve the following 2-person zero sum game with following 3×2 payoff matrix of player A.

		Player B		Row min	maximin
		B ₁	B ₂		
Player A	A ₁	9	2	2	6 Max
	A ₂	8	6	6	
	A ₃	6	4	4	

Col max

9 6

6 → Min

minimax

Minimax = 6 = Maximin
 $\therefore$ saddle point exist

optimal strategy for A is A₂
 optimal strategy for B is B₂

Value of game = 6

Ques Solve the game,

		Player B				Row min	Maximin	
		B ₁	B ₂	B ₃	B ₄			
Player A	A ₁	-5	2	0	7	-5	4 Max	
	A ₂	5	6	(4)	8	(4)		
	A ₃	4	0	2	-3	-3		
Col max		5	6	(4)	8	4 Min		
Minimax								

Minimax = 4 = Maximin
 $\therefore$ saddle point exist.

optimum strategy for A = A₂ $V=4$
 for B = B₃

Ques Solve the game.

		Player B			Row min	
		I	II	III		
A	I	2	(-1)	8	(-1)	-1 Max
	II	-4	-3	4	-4	
	III	-8	-4	0	-8	
	IV	1	-6	-2	-6	
col max		2	(-1)	8	Min -1	

Maximin = -1 = Min max

optimal strategy for A = I
 for B = II
 $V = -1$

Ques Find range of p & q that will make the payoff matrix given below.

		B			Row min	
		I	II	III		
A	I	2	4	7	2	7 Max
	II	10	7	2	7	
	III	4	p	8	4	
Cmax		10	7	8	7 min	

Maxmin = 7

Maximin = Minmax = 7
 $\therefore$ saddle pt. exist.
 Value of game = 7.
 optimal solⁿ = (II, II)

$$7 \leq q \leq 8$$

$$4 \leq p \leq 7$$

Ques Solve the following 2-person zero sum game with following 3×2 payoff matrix of player A.

		Player B			
		B ₁	B ₂	Row min	
Player A	A ₁	9	2	2	} 6 Max
	A ₂	8	6	6	
	A ₃	6	4	4	
		Col max	9 6		
		minimax	6 → Min		

Min max = 6 = Max min
 $\therefore$ saddle point exist

optimal strategy for A is A₂
 optimal strategy for B is B₂

value of game = 6

Ques Solve the game,

		Player B				
		B ₁	B ₂	B ₃	B ₄	Row min
Player A	A ₁	-5	2	0	7	-5
	A ₂	5	6	4	8	4
	A ₃	4	0	2	-3	-3
		Col max	5 6 4 8			
		minimax	4 → Min			

Min max = 4 = Max min
 $\therefore$ saddle point exist

optimum strategy for A = A₂ V = 4
 for B = B₃

Ques Solve the game.

		B			
		I	II	III	Row min
A	I	2	-1	8	-1
	II	-4	-3	4	-4
	III	-8	-4	0	-8
	IV	1	-6	-2	-6
		Col max	2 -1 8		
			Min = -1		

Maxmin = -1 = Min max

optimal strategy for A = I
 for B = II
 V = -1

Ques Find range of p & q that will make the payoff matrix given below.

		B			
		I	II	III	Row min
A	I	2	4	7	2
	II	10	7	9	7
	III	4	p	8	4
		Col max	10 7 8		
			7 → Min		

Maxmin = Min max = 7
 $\therefore$ saddle pt. exist
 optimal solⁿ = (II, II)

$$7 \leq q \leq 8$$

$$4 \leq p \leq 7$$

Game Without Saddle Point (Mixed Strategy)

A mixed strategy game can be solved by following method:

- ① Algebraic method
- ② Graphical method
- ③ Linear Programming Method

Algebraic Method

Consider the 2-person zero-sum game with following payoff matrix.

		Player B	
		I	II
Player A	I	a_{11}	a_{12}
	II	a_{21}	a_{22}

Let x_1, x_2 be prob. of using 1st & 2nd strategies by player A s.t. $x_1 + x_2 = 1$
 $\Rightarrow x_2 = 1 - x_1$

& let y_1, y_2 be prob. of using 1st & 2nd strategies by player B s.t. $y_1 + y_2 = 1$
 $\Rightarrow y_2 = 1 - y_1$

$$x_1 = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

$$y_1 = \frac{a_{22} - a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

$$V = \frac{a_{11}a_{22} - a_{21}a_{12}}{(a_{11} + a_{22}) - (a_{21} + a_{12})}$$

Ques

Solve the following game whose pay-off matrix is

		Player B		
Player A		3	-2	$3/8$
		2	5	$5/8$
		7	1	
		8	8	

optimal strategy of A = $\left(\frac{3}{8}, \frac{5}{8}\right)$

" " of B = $\left(\frac{7}{8}, \frac{1}{8}\right)$

$$V = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} \quad \text{or} \quad V = \frac{3 \times 3 + 2 \times 5}{8} = \frac{9 + 10}{8} = \frac{19}{8}$$

Ques

Find out the value of game

		B ₁	B ₂	
A	A ₁	4	-4	8/16 = 1/2
	A ₂	-4	4	8/16 = 1/2
		8	8	
		16	16	
		= 1/2	= 1/2	

optimal strategy of A = $\left(\frac{1}{2}, \frac{1}{2}\right) = 0.5$ of B

$$V = 4 \times \frac{1}{2} - 4 \times \frac{1}{2} = 0$$

∴ Fair Game

Ques

In game of matching coin, Player A wins Rs 8/- if both coin show heads & Rs 1/- if both coin show Tails, Player B wins Rs 3/- when coin do not match. Give the choice of being Player A or B, which would you choose and what would be your strategy.

		Player B		
		H	T	
Player A	H	8	-3	4/15
	T	-3	1	11/15
		4	11	
		15	15	

optimal strategy for A = $(\frac{4}{15}, \frac{11}{15})$
for B = $(\frac{4}{15}, \frac{11}{15})$

$$V = \frac{8 \times 4}{15} - \frac{3 \times 11}{15} = \frac{32}{15} - \frac{33}{15}$$

$$V = \frac{-1}{15}$$

Dominance Rule :

(i) Row Dominance

		B		
		C ₁	C ₂	C ₃
A	R ₁	4	6	-5
	R ₂	5	6	-3

Delete R₁

(ii) Column Dominance

		C	
		C ₁	C ₂
R	R ₁	4	2
	R ₂	0	-1
	R ₃	-3	-3

Delete C₁

When comparison becomes impossible, we take average of 2 rows & compare with remaining rows → same for column.

Modified Row Dominance

		B	
		B ₁	B ₂
A	A ₁	2	4
	A ₂	5	0
	A ₃	0	8
		2.5	4

Delete A₁

Modified Column Dominance

		B		
		B ₁	B ₂	B ₃
A	A ₁	8	15	1
	A ₂	1	-1	4
				15

Delete B₁

Ques

Solve the following game using the Dominance concept.

		Player B		
		B ₁	B ₂	B ₃
Player A	A ₁	4	5	8
	A ₂	6	4	6
	A ₃	4	2	4

Saddle pt doesn't exist

Step 1

Delete A₃ b'coz each element of A₂ is greater than or equal to each element of A₃

		Player B		
		B ₁	B ₂	B ₃
Player A	A ₁	4	5	8
	A ₂	6	4	6

Step 2

Delete B₃ b'coz each element of B₂ is greater than or equal to each element of B₃

Player B

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		B ₁	B ₂	
Player A	A ₁	4	5	2/3
	A ₂	6	4	1/3
		1	2	
		3	3	

optimal strategy for A = $(\frac{2}{3}, \frac{1}{3}, 0)$
 for B = $(\frac{1}{3}, \frac{2}{3}, 0)$

$$V = 2 \times \frac{4}{3} + 6 \times \frac{1}{3} = \frac{8}{3} + \frac{6}{3} = \frac{14}{3}$$

Ques Solve the following 2-person zero sum game based on the concept of Dominance.

		B			
		I	II	III	Row min
A	I	-4	6	3	-4
	II	-3	-3	4	-3
	III	2	-3	4	-3
	Col max	2	6	4	

2 min

Minmax $\neq$ Maxmin
 $\therefore$ No saddle point.

Step 1: Delete row II b'coz each element of row III is greater or equal to row II

		B		
		I	II	III
A	I	-4	6	3
	III	2	-3	4

Step 2: Delete column III b'coz each element of col III is greater or equal to col I.

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		I	II	
A	I	-4	6	5/15 $\rightarrow$ 1/3
	III	2	-3	10/15 $\rightarrow$ 2/3
		9	6	
		15	15	
		$\frac{3}{5}$	$\frac{2}{5}$	

Optimum strategy for A = $(\frac{1}{3}, 0, \frac{2}{3})$
 for B = $(\frac{3}{5}, \frac{2}{5}, 0)$

$$V = \frac{1}{3} \times -4 + \frac{4}{3} = \frac{-4}{3} + \frac{4}{3} = 0$$

$\therefore$ Fair Game

Ques Using the Dominance concept, obtain the optimal strategy.

		B					
		I	II	III	IV	V	Row min
A	I	2	4	3	8	4	2
	II	5	6	3	7	8	3
	III	6	7	9	8	7	6
	IV	4	2	8	4	3	2
	Col max	6	7	9	8	8	

6 Min

Maximum = Minmax
 $\therefore$ Saddle point exist

Step 1: Delete row II b'coz each element of row III is greater or equal to row I

Step 2: Delete row IV b'coz each element of row III is greater than or equal to row IV.

		B				
		I	II	III	IV	V
A	II	5	6	3	7	8
	III	6	7	9	8	7

Step 3: Delete Col^{III} b'coz each element of col I is greater or equal to col I

		B			
		I	III	IV	V
A	II	5	3	7	8
	III	6	9	8	7

Step 4: Delete Col IV b'coz each element of col IV is greater or equal to col I

		B		
		I	III	V
A	II	5	3	7
	III	6	9	8

Since saddle point exist

$\therefore$ ~~Value~~ $V = 6$

Optimal strategy (III, I)

Prob. for Player A = (0, 0, 1, 0)

Prob. for Player B = (1, 0, 0, 0, 0)

Ques Deduce the following game using the concept of Dominance & solve it.

		B				Row min
		I	II	III	IV	
A	II	3	2	4	0	0
	III	4	2	4	0	2
	IV	0	4	0	8	0
	Col Max	4	4	4	8	

Max min $\neq$ Minimax

$\therefore$ no saddle point

Step 1: Delete row I b'coz each element of row III is greater than or equal to row I.

		B			
		I	II	III	IV
A	II	3	2	4	0
	III	4	2	4	0
	IV	0	4	0	8

Step 2: Delete row II b'coz each element of row III is greater or equal to each element of row II.

		B			
		I	II	III	IV
A	III	4	2	4	0
	IV	0	4	0	8

Step 3: Delete Col I

		B		
		II	III	IV
A	III	2	4	0
	IV	4	0	8

Step 4: Delete Col II

		B	
		III	IV
A	III	4	0
	IV	0	8

$\frac{8}{12} \rightarrow \frac{2}{3}$
 $\frac{4}{12} \rightarrow \frac{1}{3}$

value of game = $\frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})}$
 $= \frac{32-0}{12-0} = \frac{16}{6} = \frac{8}{3} = V$

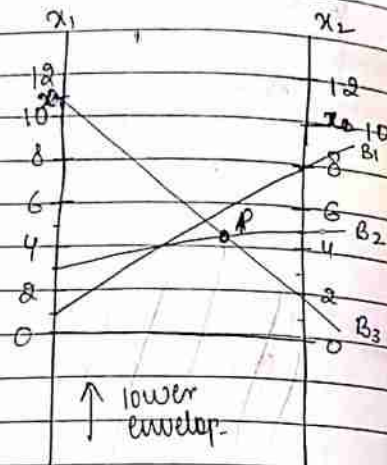
optimum strategy for A = $(0, 0, \frac{2}{3}, \frac{1}{3})$ for B

Graphical Method : Useful for $2 \times M$ & $N \times 2$ games

Ques Solve the following 2×3 game by Graphical method.

		B		
		I	II	III
A	x_1	1	3	11
	x_2	8	5	2

$E(x_1) = -7x_1 + 8$
 $E(x_2) = -2x_1 + 5$
 $E(x_3) = 9x_1 + 2$
 P is the point of maximum in the lower envelop.



		B	
		II	III
A	I	3	11
	II	5	2
		9	2
		11	11

$A \rightarrow (\frac{3}{11}, \frac{8}{11})$

$B \rightarrow (0, \frac{9}{11}, \frac{2}{11})$

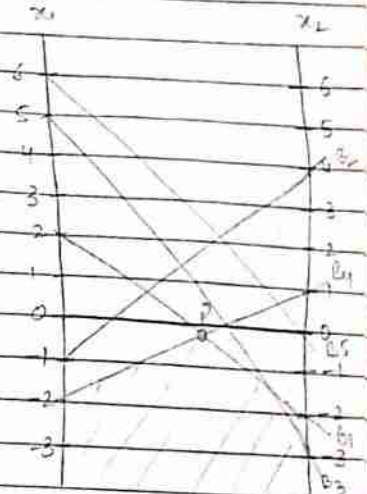
$V = \frac{3 \times 3}{11} + \frac{5 \times 8}{11} = \frac{9}{11} + \frac{40}{11} = \frac{49}{11}$

Ques Reduce the following $2 \times n$ game to 2×2 by Graphical method & solve it.

		B				
		I	II	III	IV	V
A	I	2	-1	5	-2	6
	II	-2	4	-3	1	0

$E(x_1) = 4x_1 - 2$
 $E(x_2) = -5x_1 + 4$
 $E(x_3) = 8x_1 - 3$
 $E(x_4) = -3x_1 + 1$
 $E(x_5) = 6x_1 + 0$

P is the point of maximum in the lower envelop.



		B	
		I	IV
A	I	2	-2
	II	-2	1
		3/7	4/7

$A \rightarrow (\frac{3}{7}, \frac{4}{7})$ $B \rightarrow (\frac{3}{7}, 0, 0, \frac{4}{7}, 0)$

$V = \frac{6}{7} - \frac{8}{7} = -\frac{2}{7}$

Ques Solve the following game using Graphical Method.

		B		$E(y_i)$
		B_1	B_2	
A	y_1	-6	7	$E(y_1) = -13y_1 + 7$
	y_2	4	-5	$E(y_2) = 9y_1 - 5$
	y_3	-1	-2	$E(y_3) = -y_1 - 2$
	y_4	-2	5	$E(y_4) = -7y_1 + 5$
	y_5	7	-6	$E(y_5) = 13y_1 - 6$

P is the lowest point in the upper envelop.

	B ₁	B ₂	
A ₄	-2	5	13/20
A ₅	7	-6	7/20
	11/20	9/20	

~~A₂~~

$$A \rightarrow (0, 0, 0, \frac{13}{20}, \frac{7}{20})$$

$$B \rightarrow (\frac{11}{20}, \frac{9}{20})$$

$$V = \frac{-26}{20} + \frac{49}{20} = \frac{23}{20}$$

Ques Use the dominance rule to reduce the following game in either 2xM or Nx2 game & then solve by graphical method.

		B				
		I	II	III	IV	Row min
A	I	19	6	7	5	5
	II	7	3	14	6	3
	III	12	8	18	4	4
	IV	8	7	13	-1	-1
Col max		19	8	18	6	

5 max

6 min

Max min $\neq$ Min max
 $\therefore$ no saddle point

~~Delete Row IV~~

		I	II	III	IV
A	I	19	6	7	5
	II	7	3	14	6
	III	12	8	18	4

~~Delete Col I & III & Row IV~~

		I	II	IV
A	II	6	5	
	III	3	6	
	IV	8	4	

P is the lowest point in upper envelop and is the pt. of intersection of A₁ & A₂.

$$A \rightarrow (\frac{3}{4}, \frac{1}{4}, 0, 0)$$

$$B \rightarrow (0, \frac{1}{4}, 0, \frac{3}{4})$$

		II	IV
A	I	6	5
	II	3	6

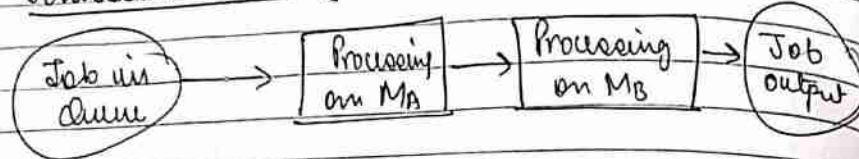
3/4
1/4

$$V = \frac{6 \times 3}{4} + \frac{3}{4} = \frac{21}{4}$$

Sequencing :

- Types of problems :
- (i) n Jobs on a single machine
 - (ii) n Jobs on two machines
 - (iii) n Jobs on three machines
 - (iv) n Jobs on m machines
 - (v) two jobs on m machine

Johnson's Rule (n Jobs on two machines)



Ques Consider the processing time (in min) of 5 Jobs each of which must go through the two machines M_1 & M_2 in order M_1, M_2 . Find the sequence for job that minimize total elapsed time. Also find ideal time for each machine. Show the sequence on Gantt chart.

Job	J_1	J_2	J_3	J_4	J_5
M_1	5	1	9	3	10
M_2	2	6	7	8	4

optimum Sequence $\rightarrow$ J_2, J_4, J_3, J_5, J_1

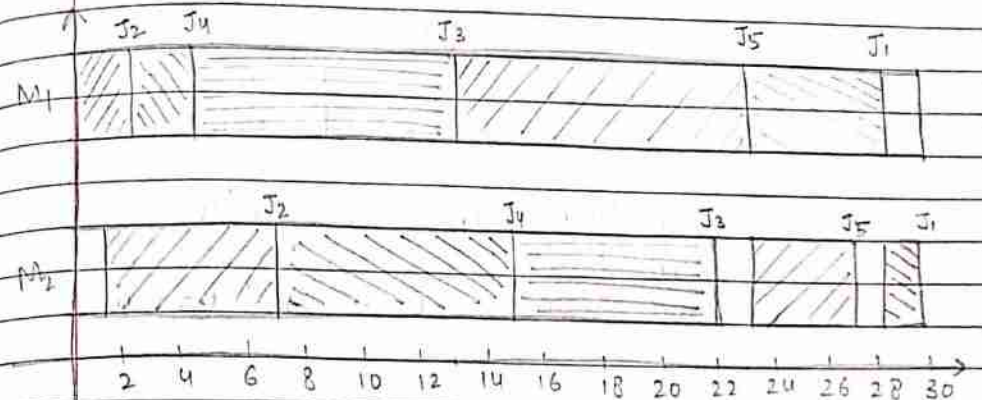
Minimum elapsed time :

Optimum sequence	Machine 1		Machine 2	
	Time IN	Time OUT	Time IN	Time OUT
2	0	1	1	7
4	1	4	7	15
3	4	13	15	22
5	13	23	23	27
1	23	28	28	30

Total time elapsed = 30 hr

Ideal time for $M_1 = (30 - 28) = 2 \text{ hr}$

Ideal time for $M_2 = (1 + 1 + 1) = 3 \text{ hr}$



Gantt Chart

Ques In a factory there are 6 jobs to be performed, each of which should go through two machines A & B in order A B. The processing time (in min) for jobs are given below. Find the sequence for job that minimize total elapsed time. Also find ideal time for

each machine.

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Job	1	2	3	4	5	6	
A	1	3	8	5	6	3	26
B	5	6	3	2	2	10	28

choose which difference is max.

optimum sequence:

1	6	2	3	4	5
---	---	---	---	---	---

Min. Time elapsed: 2

Optimum Sequence	Machine A		Machine B	
	Time IN	Time OUT	Time IN	Time OUT
1	0	1	1	6
6	1	4	6	16
2	4	7	16	22
3	7	15	22	25
4	15	20	25	27
5	20	26	27	29

Total time elapsed = 29 min

Total time for A = $(29 - 26) = 3$ min

Total time for B = ~~28~~ 1 min